ISOMETRIC IMMERSIONS INTO $\mathbb{S}^n \times \mathbb{R}$ AND $\mathbb{H}^n \times \mathbb{R}$ AND APPLICATIONS TO MINIMAL SURFACES

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Abstract. We give a necessary and sufficient condition for an n-dimensional Riemannian manifold to be isometrically immersed in $\mathbb{S}^n \times \mathbb{R}$ or $\mathbb{H}^n \times \mathbb{R}$ in terms of its first and second fundamental forms and of the projection of the vertical vector field on its tangent plane. We deduce the existence of a one-parameter family of isometric minimal deformations of a given minimal surface in $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{H}^2 \times \mathbb{R}$, obtained by rotating the shape operator.

1. Introduction

It is well known that the first and second fundamental forms of a hypersurface of a Riemannian manifold satisfy two compatibility equations called the Gauss and Codazzi equations. More precisely, let $\bar{\mathcal{V}}$ be an orientable Riemannian manifold of dimension $n+1$ and $\mathcal{V}$ a submanifold of $\bar{\mathcal{V}}$ of dimension n. Let ∇ (respectively, $\bar{\nabla}$) be the Riemannian connection of $\mathcal{V}$ (respectively, $\bar{\mathcal{V}}$), R (respectively, $\bar{\mathrm{R}}$) the Riemann curvature tensor of $\mathcal{V}$ (respectively, $\bar{\mathcal{V}}$), i.e.,

$$\mathrm{R}(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z,$$

and S the shape operator of $\mathcal{V}$ associated to its unit normal N, i.e., $\mathrm{S}X = -\bar{\nabla}_X N$. Then the following equations hold for all vector fields X, Y, Z on $\mathcal{V}$:

$$\mathrm{R}(X,Y)Z - \bar{\mathrm{R}}(X,Y)Z = \langle \mathrm{S}X, Z\rangle \mathrm{S}Y - \langle \mathrm{S}Y, Z\rangle \mathrm{S}X,$$
$$\nabla_X \mathrm{S}Y - \nabla_Y \mathrm{S}X - \mathrm{S}[X,Y] = \bar{\mathrm{R}}(X,Y)N.$$

These are respectively the Gauss and Codazzi equations.

In the case where $\bar{\mathcal{V}}$ is a space form, i.e., the sphere $\mathbb{S}^{n+1}$, the Euclidean space $\mathbb{R}^{n+1}$ or the hyperbolic space $\mathbb{H}^{n+1}$, these equations become the following:

$$\begin{aligned}\langle \mathrm{R}(X,Y)Z, W\rangle - \kappa(\langle X,Z\rangle\langle Y,W\rangle - \langle Y,Z\rangle\langle X,W\rangle) \\ = \langle \mathrm{S}X,Z\rangle\langle \mathrm{S}Y,W\rangle - \langle \mathrm{S}X,W\rangle\langle \mathrm{S}Y,Z\rangle,\end{aligned} \tag{1}$$

$$\nabla_X \mathrm{S}Y - \nabla_Y \mathrm{S}X - \mathrm{S}[X,Y] = 0, \tag{2}$$

where κ is the sectional curvature of $\bar{\mathcal{V}}$, i.e., $\kappa = 1, 0, -1$ for $\mathbb{S}^{n+1}$, $\mathbb{R}^{n+1}$ and $\mathbb{H}^{n+1}$ respectively. Thus the Gauss and Codazzi equations only involve the first and second fundamental forms of $\mathcal{V}$; they are defined *intrinsicly* on $\mathcal{V}$ (as soon as we know S). This comes from the fact that these ambiant spaces are isotropic. Moreover, in this case the Gauss and Codazzi equations are also sufficient conditions for an n-dimensional simply connected manifold to be immersed into $\bar{\mathcal{V}}$ with given first and second fundamental forms: if $\mathcal{V}$ is a Riemannian manifold endowed with a field S of

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symmetric operators $\mathrm{S}_y : \mathrm{T}_y\mathcal{V} \to \mathrm{T}_y\mathcal{V}$ such that (1) and (2) hold (where R denotes the Riemann curvature tensor of $\mathcal{V}$), then there exists an isometric immersion from $\mathcal{V}$ into $\bar{\mathcal{V}}$ with S as shape operator. The reader can refer to [Car92], and also to [Ten71] for a proof in the case of $\mathbb{R}^{n+1}$.

In the case of a general manifold $\bar{\mathcal{V}}$, the Gauss and Codazzi equations are not defined intrinsicly on $\mathcal{V}$, since the Riemann curvature tensor of the ambiant space $\bar{\mathcal{V}}$ is involved. Yet, in the case where $\bar{\mathcal{V}} = \mathbb{S}^n \times \mathbb{R}$ or $\bar{\mathcal{V}} = \mathbb{H}^n \times \mathbb{R}$, these equations are well defined as soon as we know:

(1) the projection T of the vertical vector $\frac{\partial}{\partial t}$ (corresponding to the factor $\mathbb{R}$) onto the tangent space of $\mathcal{V}$,
(2) the normal component ν of $\frac{\partial}{\partial t}$, i.e., $\nu = \langle N, \frac{\partial}{\partial t}\rangle$.

Indeed, the Gauss and Codazzi equations become the following:

$$\begin{aligned} \mathrm{R}(X,Y)Z \quad = \quad & \langle \mathrm{S}X, Z\rangle \mathrm{S}Y - \langle \mathrm{S}Y, Z\rangle \mathrm{S}X \\ & +\kappa(\langle X, Z\rangle Y - \langle Y, Z\rangle X - \langle Y, T\rangle\langle X, Z\rangle T \\ & -\langle X, T\rangle\langle Z, T\rangle Y + \langle X, T\rangle\langle Y, Z\rangle T + \langle Y, T\rangle\langle Z, T\rangle X), \end{aligned}$$

$$\nabla_X \mathrm{S}Y - \nabla_Y \mathrm{S}X - \mathrm{S}[X,Y] = \kappa\nu(\langle Y, T\rangle X - \langle X, T\rangle Y),$$

where $\kappa = 1$ and $\kappa = -1$ for $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ respectively.

The Gauss equation can be formulated in the following equivalent way: the sectional curvature $K(P)$ (for the metric of $\mathcal{V}$) of every plane $P \subset \mathrm{T}\mathcal{V}$ satisfies

$$K(P) = \det \mathrm{S}_P + \kappa(1 - ||T_P||^2)$$

where S_P is the restriction of S on P and T_P the orthogonal projection of T on P.

The first aim of this paper is to give a necessary and sufficient condition in order that a Riemannian manifold with a symmetric operator S can be isometrically immersed into $\mathbb{S}^n \times \mathbb{R}$ or $\mathbb{H}^n \times \mathbb{R}$ with S as shape operator. More precisely, we prove the following theorem.

Theorem (theorem 3.3)**.** *Let $\mathcal{V}$ be a simply connected Riemannian manifold of dimension n, $\mathrm{d}s^2$ its metric (which we also denote by $\langle\cdot,\cdot\rangle$) and ∇ its Riemannian connection. Let S be a field of symmetric operators $\mathrm{S}_y : \mathrm{T}_y\mathcal{V} \to \mathrm{T}_y\mathcal{V}$, T a vector field on $\mathcal{V}$ and ν a smooth function on $\mathcal{V}$ such that $||T||^2 + \nu^2 = 1$.*

Let $\mathbb{M}^n = \mathbb{S}^n$ or $\mathbb{M}^n = \mathbb{H}^n$. Assume that $(\mathrm{d}s^2, \mathrm{S}, T, \nu)$ satisfies the Gauss and Codazzi equations for $\mathbb{M}^n \times \mathbb{R}$ and the following equations:

$$\nabla_X T = \nu \mathrm{S}X, \quad \mathrm{d}\nu(X) = -\langle \mathrm{S}X, T\rangle.$$

Then there exists an isometric immersion $f : \mathcal{V} \to \mathbb{M}^n \times \mathbb{R}$ such that the shape operator with respect to the normal N associated to f is

$$\mathrm{d}f \circ \mathrm{S} \circ \mathrm{d}f^{-1}$$

and such that

$$\frac{\partial}{\partial t} = \mathrm{d}f(T) + \nu N.$$

Moreover the immersion is unique up to a global isometry of $\mathbb{M}^n \times \mathbb{R}$ preserving the orientations of both $\mathbb{M}^n$ and $\mathbb{R}$.

The two additional conditions come from the fact that the vertical vector field $\frac{\partial}{\partial t}$ is parallel.

The method to prove this theorem is similar to that of Tenenblat ([Ten71]): it is based on differential forms, moving frames and integrable distributions.

This work was motivated by the study of minimal surfaces in $\mathbb{S}^2\times\mathbb{R}$ and $\mathbb{H}^2\times\mathbb{R}$. There were many recent developments in the theory of these surfaces. Rosenberg ([Ros02b]) studied the geometry of minimal surfaces in $\mathbb{S}^2\times\mathbb{R}$, and more generally in $M\times\mathbb{R}$ where M is a surface of non-negative curvature. Nelli and Rosenberg ([NR02]) studied minimal surfaces in $\mathbb{H}^2\times\mathbb{R}$ and proved a Jenkins-Serrin theorem. Hauswirth ([Hau03]) constructed many examples in $\mathbb{H}^2\times\mathbb{R}$. Meeks and Rosenberg ([MR03]) initiated the theory of minimal surfaces in $M\times\mathbb{R}$ where M is a compact surface. Recently, Abresch and Rosenberg ([AR03]) extended the notion of holomorphic Hopf differential to constant mean curvature surfaces in $\mathbb{S}^2\times\mathbb{R}$ and $\mathbb{H}^2\times\mathbb{R}$; using this holomorphic differential, they proved that all immersed constant mean curvature spheres are embedded and rotational.

In this paper, we use our theorem 3.3 to prove the existence of a one-parameter family of isometric minimal deformations of a given minimal surface in $\mathbb{S}^2\times\mathbb{R}$ or $\mathbb{H}^2\times\mathbb{R}$. This family is obtained by rotating the shape operator; hence it is the analog of the associate family of a minimal surface in $\mathbb{R}^3$. This is the following theorem.

Theorem (theorem 4.2)**.** *Let Σ be a simply connected Riemann surface and $x:\Sigma\to\mathbb{M}^2\times\mathbb{R}$ a conformal minimal immersion. Let N be the induced normal. Let S be the symmetric operator on Σ induced by the shape operator of $x(\Sigma)$. Let T be the vector field on Σ such that $\mathrm{d}x(T)$ is the projection of $\frac{\partial}{\partial t}$ onto $\mathrm{T}(x(\Sigma))$. Let $\nu=\langle N,\frac{\partial}{\partial t}\rangle$.*

Let $z_0\in\Sigma$. Then there exists a unique family $(x_\theta)_{\theta\in\mathbb{R}}$ of conformal minimal immersions $x_\theta:\Sigma\to\mathbb{M}^2\times\mathbb{R}$ such that:

1. *$x_\theta(z_0)=x(z_0)$ and $(\mathrm{d}x_\theta)_{z_0}=(\mathrm{d}x)_{z_0}$,*
2. *the metrics induced on Σ by x and x_θ are the same,*
3. *the symmetric operator on Σ induced by the shape operator of $x_\theta(\Sigma)$ is $e^{\theta\mathrm{J}}\mathrm{S}$,*
4. *$\frac{\partial}{\partial t}=\mathrm{d}x_\theta(e^{\theta\mathrm{J}}T)+\nu N_\theta$ where N_θ is the unit normal to x_θ.*

Moreover we have $x_0=x$ and the family (x_θ) is continuous with respect to θ.

In particular taking $\theta=\frac{\pi}{2}$ defines a conjugate surface; the geometric properties of conjugate surfaces in $\mathbb{M}^2\times\mathbb{R}$ and in $\mathbb{R}^3$ are similar. Finally, we give examples of conjugate surfaces. In $\mathbb{S}^2\times\mathbb{R}$, we show that helicoids and unduloids are conjugate. In $\mathbb{H}^2\times\mathbb{R}$, we show that helicoids are conjugated to catenoids or to minimal surfaces foliated by horizontal curves of constant curvature belonging to the Hauswirth family (see [Hau03]).

2. Preliminaries

Notations. In this paper we will use the following index conventions: Latin letters i, j, etc, denote integers between 1 and n, Greek letters α, β, etc, denote integers between 0 and $n+1$. For example, the notation $A^i_j=B^i_j$ means that this relation holds for all integers i, j between 1 and n, the notation $\sum_\alpha C_\alpha$ means $C_0+C_1+\cdots+C_{n+1}$.

The set of vector fields on a Riemannian manifold $\mathcal{V}$ will be denoted by $\mathfrak{X}(\mathcal{V})$.

We denote by $\frac{\partial}{\partial t}$ the unit vector giving the orientation of $\mathbb{R}$ in $\mathbb{M}^n\times\mathbb{R}$; we call it the vertical vector.

2.1. **The compatibility equations in $\mathbb{M}^n\times\mathbb{R}$.**

Let $\mathbb{M}^n=\mathbb{S}^n$ or $\mathbb{M}^n=\mathbb{H}^n$; in the first case we set $\kappa=1$ and in the second case we set $\kappa=-1$. Let $\bar{\mathrm{R}}$ be the

Riemann curvature tensor of $\mathbb{M}^n\times\mathbb{R}$. Let $\mathcal{V}$ be an oriented hypersurface of $\mathbb{M}^n\times\mathbb{R}$ and N the unit normal to $\mathcal{V}$.

Proposition 2.1. *For $X,Y,Z,W\in\mathfrak{X}(\mathcal{V})$ we have*

$$\begin{aligned}\langle\bar{\mathrm{R}}(X,Y)Z,W\rangle &= \kappa(\langle X,Z\rangle\langle Y,W\rangle-\langle Y,Z\rangle\langle X,W\rangle\\ &\quad -\langle Y,T\rangle\langle W,T\rangle\langle X,Z\rangle-\langle X,T\rangle\langle Z,T\rangle\langle Y,W\rangle\\ &\quad +\langle X,T\rangle\langle W,T\rangle\langle Y,Z\rangle+\langle Y,T\rangle\langle Z,T\rangle\langle X,W\rangle),\end{aligned}$$

$$\langle\bar{\mathrm{R}}(X,Y)N,Z\rangle=\kappa\nu(\langle X,Z\rangle\langle Y,T\rangle-\langle Y,Z\rangle\langle X,T\rangle),$$

where

$$\nu=\left\langle N,\frac{\partial}{\partial t}\right\rangle$$

and T is the projection of $\frac{\partial}{\partial t}$ on $\mathrm{T}\mathcal{V}$, i.e.,

$$T=\frac{\partial}{\partial t}-\nu N.$$

Proof. Any vector field on $\mathbb{M}^n\times\mathbb{R}$ can be written $X(m,t)=(X^t_{\mathbb{M}^n}(m),X^m_{\mathbb{R}}(t))$ where, for each $t\in\mathbb{R}$, $X^t_{\mathbb{M}^n}$ is a vector field on $\mathbb{M}^n$, and, for each $m\in\mathbb{M}^n$, $X^m_{\mathbb{R}}$ is a vector field on $\mathbb{R}$. Then for $X,Y,Z,W\in\mathfrak{X}(\mathbb{M}^n\times\mathbb{R})$ we have

$$\begin{aligned}\langle\bar{\mathrm{R}}(X,Y)Z,W\rangle &= \langle\bar{\mathrm{R}}_{\mathbb{M}^n}(X_{\mathbb{M}^n},Y_{\mathbb{M}^n})Z_{\mathbb{M}^n},W_{\mathbb{M}^n}\rangle\\ &= \kappa(\langle X_{\mathbb{M}^n},Z_{\mathbb{M}^n}\rangle\langle Y_{\mathbb{M}^n},W_{\mathbb{M}^n}\rangle-\langle Y_{\mathbb{M}^n},Z_{\mathbb{M}^n}\rangle\langle X_{\mathbb{M}^n},W_{\mathbb{M}^n}\rangle).\end{aligned}$$

We have $X_{\mathbb{M}^n}=X-\left\langle X,\frac{\partial}{\partial t}\right\rangle\frac{\partial}{\partial t}$. Thus, if $X\in\mathrm{T}\mathcal{V}$, we have $X_{\mathbb{M}^n}=X-\langle X,T\rangle\frac{\partial}{\partial t}$, and similar expressions for $Y,Z,W\in\mathrm{T}\mathcal{V}$. A computation gives the expected formula for $\langle\bar{\mathrm{R}}(X,Y)Z,W\rangle$.

Finally we have $N_{\mathbb{M}^n}=N-\nu\frac{\partial}{\partial t}$, so a computation gives the expected formula for $\langle\bar{\mathrm{R}}(X,Y)N,Z\rangle$. □

Using the fact that the vector field $\frac{\partial}{\partial t}$ is parallel, we obtain the following equations.

Proposition 2.2. *For $X\in\mathfrak{X}(\mathcal{V})$ we have*

$$\nabla_XT=\nu\mathrm{S}X,\quad \mathrm{d}\nu(X)=-\langle\mathrm{S}X,T\rangle.$$

Proof. We have $\frac{\partial}{\partial t}=T+\nu N$ and $\bar\nabla_X\frac{\partial}{\partial t}=0$. Thus we get

$$0=\bar\nabla_XT+(\mathrm{d}\nu(X))N+\nu\bar\nabla_XN=\nabla_XT+\langle\mathrm{S}X,T\rangle N+(\mathrm{d}\nu(X))N-\nu\mathrm{S}X.$$

Taking the tangential and the normal components in this equality, we obtain the expected formulas. □

Remark 2.3. In the case of an orthonormal pair (X,Y) we get

$$\langle\bar{\mathrm{R}}(X,Y)X,Y\rangle=\kappa(1-\langle Y,T\rangle^2-\langle X,T\rangle^2).$$

The reader can also refer to section 3.2 in [AR03].

2.2. **Moving frames.** In this section we introduce some material about the technique of moving frames. The reader can also refer to [Ros02a].

Let $\mathcal{V}$ be a Riemannian manifold of dimension n, ∇ its Levi-Civita connection, and R the Riemannian curvature tensor. Let S be a field of symmetric operators $\mathrm{S}_y : \mathrm{T}_y\mathcal{V} \to \mathrm{T}_y\mathcal{V}$. Let $(e_1, \dots, e_n)$ be a local orthonormal frame on $\mathcal{V}$ and $(\omega^1, \dots, \omega^n)$ the dual basis of $(e_1, \dots, e_n)$, i.e.,

$$\omega^i(e_k) = \delta^i_k.$$

We also set

$$\omega^{n+1} = 0.$$

We define the forms ω^i_j, ω^{n+1}_j, ω^i_{n+1} and ω^{n+1}_{n+1} on $\mathcal{V}$ by

$$\omega^i_j(e_k) = \langle \nabla_{e_k} e_j, e_i\rangle, \quad \omega^{n+1}_j(e_k) = \langle \mathrm{S}e_k, e_j\rangle,$$

$$\omega^j_{n+1} = -\omega^{n+1}_j, \quad \omega^{n+1}_{n+1} = 0.$$

Then we have

$$\nabla_{e_k} e_j = \sum_i \omega^i_j(e_k) e_i, \quad \mathrm{S}e_k = \sum_j \omega^{n+1}_j(e_k) e_j.$$

Finally we set $R^i_{klj} = \langle \mathrm{R}(e_k, e_l)e_j, e_i\rangle$.

Proposition 2.4. *We have the following formulas:*

$$\mathrm{d}\omega^i + \sum_p \omega^i_p \wedge \omega^p = 0, \tag{3}$$

$$\sum_p \omega^{n+1}_p \wedge \omega^p = 0, \tag{4}$$

$$\mathrm{d}\omega^i_j + \sum_p \omega^i_p \wedge \omega^p_j = -\frac{1}{2}\sum_k\sum_l R^i_{klj}\omega^k \wedge \omega^l, \tag{5}$$

$$\mathrm{d}\omega^{n+1}_j + \sum_p \omega^{n+1}_p \wedge \omega^p_j = \frac{1}{2}\sum_k\sum_l \langle \nabla_{e_k}\mathrm{S}e_l - \nabla_{e_l}\mathrm{S}e_k - \mathrm{S}[e_k, e_l], e_j\rangle \omega^k \wedge \omega^l. \tag{6}$$

Proof. These are well known formulas. However, since our conventions slightly differ from those of [Ten71] and [Ros02a], we give a proof for sake of clarity.

We have $\mathrm{d}\omega^i(e_p, e_q) = -\omega^i([e_p, e_q]) = -\omega^i(\nabla_{e_p} e_q - \nabla_{e_q} e_p) = -\omega^i_q(e_p) + \omega^i_p(e_q)$ and $\sum_k \omega^i_k \wedge \omega^k(e_p, e_q) = \omega^i_q(e_p) - \omega^i_p(e_q)$, so (3) is proved. And we have $\sum_k(\omega^{n+1}_k \wedge \omega^k)(e_p, e_q) = \omega^{n+1}_q(e_p) - \omega^{n+1}_p(e_q) = \langle \mathrm{S}e_p, e_q\rangle - \langle \mathrm{S}e_q, e_p\rangle = 0$, so (4) is proved.

We have $\omega^i_j = \sum_k \langle e_i, \nabla_{e_k} e_j\rangle \omega^k$, so

$$\begin{aligned} \mathrm{d}\omega^i_j &= \sum_k\sum_l e_l\langle e_i, \nabla_{e_k} e_j\rangle \omega^l \wedge \omega^k + \sum_k \langle e_i, \nabla_{e_k} e_j\rangle \mathrm{d}\omega^k \\ &= \sum_k\sum_l (\langle \nabla_{e_l} e_i, \nabla_{e_k} e_j\rangle + \langle e_i, \nabla_{e_l}\nabla_{e_k} e_j\rangle)\omega^l \wedge \omega^k \\ &\quad -\sum_k\sum_l \langle e_i, \nabla_{e_k} e_j\rangle \omega^k_l \wedge \omega^l. \end{aligned}$$

Moreover we have

$$\begin{aligned}\sum_k\sum_l\langle e_i,\nabla_{e_k}e_j\rangle\omega_l^k\wedge\omega^l &= \sum_k\sum_l\sum_q\langle e_i,\nabla_{e_k}e_j\rangle\langle e_k,\nabla_{e_q}e_l\rangle\omega^q\wedge\omega^l\\ &= \sum_l\sum_q\langle e_i,\nabla_{\nabla_{e_q}e_l}e_j\rangle\omega^q\wedge\omega^l.\end{aligned}$$

On the other hand we have

$$\begin{aligned}\sum_p\omega_p^i\wedge\omega_j^p &= \sum_k\sum_l\sum_p\langle e_i,\nabla_{e_l}e_p\rangle\langle e_p,\nabla_{e_k}e_j\rangle\omega^l\wedge\omega^k\\ &= -\sum_k\sum_l\sum_p\langle\nabla_{e_l}e_i,e_p\rangle\langle e_p,\nabla_{e_k}e_j\rangle\omega^l\wedge\omega^k\\ &= -\sum_k\sum_l\langle\nabla_{e_l}e_i,\nabla_{e_k}e_j\rangle\omega^l\wedge\omega^k.\end{aligned}$$

Thus we conclude that

$$\mathrm{d}\omega_j^i+\sum_p\omega_p^i\wedge\omega_j^p=\sum_k\sum_l\langle e_i,\nabla_{e_l}\nabla_{e_k}e_j-\nabla_{\nabla_{e_l}e_k}e_j\rangle\omega^l\wedge\omega^k.$$

Adding this equality with itself after exchanging k and l and using the fact that $\omega^k\wedge\omega^l=-\omega^l\wedge\omega^k$, we get

$$2\left(\mathrm{d}\omega_j^i+\sum_p\omega_p^i\wedge\omega_j^p\right) = \sum_k\sum_l\langle e_i,\mathrm{R}(e_k,e_l)e_j\rangle\omega^l\wedge\omega^k,$$

and finally we get (5).

We have $\omega_j^{n+1}=\sum_k\langle\mathrm{S}e_k,e_j\rangle\omega^k$, so

$$\begin{aligned}\mathrm{d}\omega_j^{n+1} &= \sum_k\sum_l e_l\langle\mathrm{S}e_k,e_j\rangle\omega^l\wedge\omega^k+\sum_k\langle\mathrm{S}e_k,e_j\rangle\mathrm{d}\omega^k\\ &= \sum_k\sum_l(\langle\nabla_{e_l}\mathrm{S}e_k,e_j\rangle+\langle\mathrm{S}e_k,\nabla_{e_l}e_j\rangle)\omega^l\wedge\omega^k-\sum_k\sum_l\langle\mathrm{S}e_k,e_j\rangle\omega_l^k\wedge\omega^l.\end{aligned}$$

Moreover we have

$$\begin{aligned}\sum_k\sum_l\langle\mathrm{S}e_k,e_j\rangle\omega_l^k\wedge\omega^l &= \sum_k\sum_l\sum_q\langle\mathrm{S}e_k,e_j\rangle\langle e_k,\nabla_{e_q}e_l\rangle\omega^q\wedge\omega^l\\ &= \sum_l\sum_q\langle\mathrm{S}e_j,\nabla_{e_q}e_l\rangle\omega^q\wedge\omega^l.\end{aligned}$$

On the other hand we have

$$\begin{aligned}\sum_p\omega_p^{n+1}\wedge\omega_j^p &= \sum_k\sum_p\langle\mathrm{S}e_k,e_p\rangle\omega^k\wedge\omega_j^p\\ &= \sum_k\sum_p\sum_l\langle\mathrm{S}e_k,e_p\rangle\langle e_p,\nabla_{e_l}e_j\rangle\omega^k\wedge\omega^l\\ &= \sum_k\sum_l\langle\mathrm{S}e_k,\nabla_{e_l}e_j\rangle\omega^k\wedge\omega^l.\end{aligned}$$

Thus we conclude that

$$\begin{aligned}
\mathrm{d}\omega_j^{n+1} + \sum_p \omega_p^{n+1} \wedge \omega_j^p &= \sum_k \sum_l (\langle \nabla_{e_l} \mathrm{S} e_k, e_j \rangle - \langle \mathrm{S} e_j, \nabla_{e_l} e_k \rangle) \omega^l \wedge \omega^k \\
&= \sum_k \sum_l \langle e_j, \nabla_{e_l} \mathrm{S} e_k - \mathrm{S} \nabla_{e_l} e_k \rangle \omega^l \wedge \omega^k.
\end{aligned}$$

Adding this equality with itself after exchanging k and l and using the fact that $\omega^k \wedge \omega^l = -\omega^l \wedge \omega^k$, we get

$$2\left(\mathrm{d}\omega_j^{n+1} + \sum_p \omega_p^{n+1} \wedge \omega_j^p\right) = \sum_k \sum_l \langle e_j, \nabla_{e_l} \mathrm{S} e_k - \nabla_{e_k} \mathrm{S} e_l - \mathrm{S}[e_l, e_k] \rangle \omega^l \wedge \omega^k,$$

and finally we get (6). □

2.3. **Some facts about hypersurfaces of $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$.** In this section we consider an orientable hypersurface $\mathcal{V}$ of $\mathbb{M}^n \times \mathbb{R}$ with $\mathbb{M}^n = \mathbb{S}^n$ or $\mathbb{M}^n = \mathbb{H}^n$.

We denote by $\mathbb{L}^p$ the p-dimensional Lorentz space, i.e., $\mathbb{R}^p$ endowed with the quadradic form

$$-(\mathrm{d}x^0)^2 + (\mathrm{d}x^1)^2 + \cdots + (\mathrm{d}x^{p-1})^2.$$

We will use the following inclusions: we have

$$\mathbb{S}^n = \{(x^0, \dots, x^n) \in \mathbb{R}^{n+1}; (x^0)^2 + \sum_i (x^i)^2 = 1\}$$

and so

$$\mathbb{S}^n \times \mathbb{R} \subset \mathbb{R}^{n+1} \times \mathbb{R} = \mathbb{R}^{n+2},$$

and we have

$$\mathbb{H}^n = \{(x^0, \dots, x^n) \in \mathbb{L}^{n+1}; -(x^0)^2 + \sum_i (x^i)^2 = -1, x^0 > 0\}$$

and so

$$\mathbb{H}^n \times \mathbb{R} \subset \mathbb{L}^{n+1} \times \mathbb{R} = \mathbb{L}^{n+2}.$$

In the case of $\mathbb{S}^n \times \mathbb{R}$ we set $\kappa = 1$ and $\mathbb{E}^{n+2} = \mathbb{R}^{n+2}$. In the case of $\mathbb{H}^n \times \mathbb{R}$ we set $\kappa = -1$ and $\mathbb{E}^{n+2} = \mathbb{L}^{n+2}$.

We denote by ∇, $\bar{\nabla}$ and $\bar{\bar{\nabla}}$ the connections of $\mathcal{V}$, $\mathbb{M}^n \times \mathbb{R}$ and $\mathbb{E}^{n+2}$ respectively, by $\bar{N}(x)$ the normal to $\mathbb{M}^n \times \mathbb{R}$ in $\mathbb{E}^{n+2}$ at a point $x \in \mathbb{M}^n \times \mathbb{R}$, i.e.,

$$\bar{N}(x) = (x^0, \dots, x^n, 0),$$

and by $N(x)$ the normal to $\mathcal{V}$ in $\mathbb{M}^n \times \mathbb{R}$ at a point $x \in \mathcal{V}$. We denote by S the shape operator of $\mathcal{V}$ in $\mathbb{M}^n \times \mathbb{R}$. The shape operator of $\mathbb{M}^n \times \mathbb{R}$ is $\bar{\mathrm{S}}X = -\kappa \mathrm{d}\bar{N}(X) = \kappa\left(-X + \left\langle X, \frac{\partial}{\partial t}\right\rangle \frac{\partial}{\partial t}\right)$; we should be careful with the sign convention in the definition of the shape operator: here we have chosen

$$\bar{\bar{\nabla}}_X Y = \bar{\nabla}_X Y + \langle \bar{\mathrm{S}}X, Y \rangle \bar{N},$$

i.e.,

$$\langle \bar{\mathrm{S}}X, Y \rangle = \kappa \langle \bar{\bar{\nabla}}_X Y, \bar{N} \rangle,$$

because in the case of $\mathbb{S}^n \times \mathbb{R}$ we have $\langle \bar{N}, \bar{N} \rangle = 1$ whereas in the case of $\mathbb{H}^n \times \mathbb{R}$ we have $\langle \bar{N}, \bar{N} \rangle = -1$.

Let $(e_1, \dots, e_n)$ be a local orthonormal frame on $\mathcal{V}$, $e_{n+1} = N$ and $e_0 = \bar{N}$ (on $\mathcal{V}$). We define the forms ω_j^i, ω_j^{n+1}, ω_{n+1}^i and ω_{n+1}^{n+1} as in section 2.2. Moreover we set

$$\omega_\gamma^0(e_k) = \langle \bar{\mathrm{S}} e_k, e_\gamma \rangle = -\kappa \langle e_k, e_\gamma \rangle + \kappa \left\langle e_k, \frac{\partial}{\partial t} \right\rangle \left\langle e_\gamma, \frac{\partial}{\partial t} \right\rangle,$$

$$\omega_0^\gamma = -\kappa \omega_\gamma^0.$$

With these definitions we have

$$\bar{\bar{\nabla}}_{e_k} e_\beta = \sum_\alpha \omega_\beta^\alpha(e_k) e_\alpha.$$

Let $(E_0, \dots, E_{n+1})$ be the canonical frame of $\mathbb{E}^{n+2}$ (with $\langle E_0, E_0 \rangle = \kappa$ and $E_{n+1} = \frac{\partial}{\partial t}$). Let $A \in \mathcal{M}_{n+2}(\mathbb{R})$ be the matrix (the indices going from 0 to $n+1$) whose columns are the coordinates of the e_β in the frame (E_α), i.e.,

$$e_\beta = \sum_\alpha A_\beta^\alpha E_\alpha.$$

Then, on the one hand we have

$$\bar{\bar{\nabla}}_{e_k} e_\beta = \sum_\alpha \mathrm{d}A_\beta^\alpha(e_k) E_\alpha,$$

and on the other hand we have

$$\bar{\bar{\nabla}}_{e_k} e_\beta = \sum_\alpha \sum_\gamma \omega_\beta^\gamma(e_k) A_\gamma^\alpha E_\alpha.$$

Thus we have

$$A^{-1} \mathrm{d}A = \Omega$$

with $\Omega = (\omega_\beta^\alpha) \in \mathcal{M}_{n+2}(\mathbb{R})$, the indices going from 0 to $n+1$.

Setting $G = \mathrm{diag}(\kappa, 1, \dots, 1) \in \mathcal{M}_{n+2}(\mathbb{R})$, we have

$$A \in \mathrm{SO}^+(\mathbb{E}^{n+2}), \quad \Omega \in \mathfrak{so}(\mathbb{E}^{n+2})$$

where $\mathrm{SO}^+(\mathbb{E}^{n+2})$ is the connected component of I_{n+2} in

$$\mathrm{SO}(\mathbb{E}^{n+2}) = \{Z \in \mathcal{M}_{n+2}(\mathbb{R}); {}^{\mathrm{t}}ZGZ = G, \det Z = 1\},$$

and where

$$\mathfrak{so}(\mathbb{E}^{n+2}) = \{H \in \mathcal{M}_{n+2}(\mathbb{R}); {}^{\mathrm{t}}HG + GH = 0\}.$$

In the case of $\mathbb{S}^n \times \mathbb{R}$ we have $\mathrm{SO}^+(\mathbb{E}^{n+2}) = \mathrm{SO}(\mathbb{R}^{n+2})$.

3. Isometric immersions into $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$

3.1. **The compatibility equations.** We consider a simply connected Riemannian manifold $\mathcal{V}$ of dimension n. Let $\mathrm{d}s^2$ be the metric on $\mathcal{V}$ (we will also denote it by $\langle \cdot, \cdot \rangle$), ∇ the Riemannian connection of $\mathcal{V}$ and R its Riemann curvature tensor. Let S be a field of symmetric operators $\mathrm{S}_y : \mathrm{T}_y\mathcal{V} \to \mathrm{T}_y\mathcal{V}$, T a vector field on $\mathcal{V}$ such that $||T|| \leqslant 1$ and ν a smooth function on $\mathcal{V}$ such that $\nu^2 \leqslant 1$.

The compatibility equations for hypersurfaces in $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ established in section 2.1 suggest to introduce the following definition.

Definition 3.1. We say that $(\mathrm{d}s^2, \mathrm{S}, T, \nu)$ satisfies the compatibility equations respectively for $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ if

$$||T||^2 + \nu^2 = 1$$

and, for all $X, Y, Z \in \mathfrak{X}(\mathcal{V})$,

$$\begin{aligned} \mathrm{R}(X,Y)Z &= \langle \mathrm{S}X, Z\rangle \mathrm{S}Y - \langle \mathrm{S}Y, Z\rangle \mathrm{S}X \\ &\quad +\kappa(\langle X,Z\rangle Y - \langle Y,Z\rangle X - \langle Y,T\rangle\langle X,Z\rangle T \\ &\quad -\langle X,T\rangle\langle Z,T\rangle Y + \langle X,T\rangle\langle Y,Z\rangle T + \langle Y,T\rangle\langle Z,T\rangle X), \end{aligned} \tag{7}$$

$$\nabla_X \mathrm{S}Y - \nabla_Y \mathrm{S}X - \mathrm{S}[X,Y] = \kappa\nu(\langle Y,T\rangle X - \langle X,T\rangle Y), \tag{8}$$

$$\nabla_X T = \nu \mathrm{S}X, \tag{9}$$

$$\mathrm{d}\nu(X) = -\langle \mathrm{S}X, T\rangle, \tag{10}$$

where $\kappa = 1$ and $\kappa = -1$ for $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ respectively.

Remark 3.2. We notice that (9) implies (10) except when $\nu = 0$ (by differentiating the identity $\langle T, T\rangle + \nu^2 = 1$ with respect to X).

3.2. **Codimension 1 isometric immersions into $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$.** In this section we will prove the following theorem.

Theorem 3.3. *Let $\mathcal{V}$ be a simply connected Riemannian manifold of dimension n, $\mathrm{d}s^2$ its metric and ∇ its Riemannian connection. Let S be a field of symmetric operators $\mathrm{S}_y : \mathrm{T}_y\mathcal{V} \to \mathrm{T}_y\mathcal{V}$, T a vector field on $\mathcal{V}$ and ν a smooth function on $\mathcal{V}$ such that $||T||^2 + \nu^2 = 1$.*

Let $\mathbb{M}^n = \mathbb{S}^n$ or $\mathbb{M}^n = \mathbb{H}^n$. Assume that $(\mathrm{d}s^2, \mathrm{S}, T, \nu)$ satisfies the compatibility equations for $\mathbb{M}^n \times \mathbb{R}$. Then there exists an isometric immersion $f : \mathcal{V} \to \mathbb{M}^n \times \mathbb{R}$ such that the shape operator with respect to the normal N associated to f is

$$\mathrm{d}f \circ \mathrm{S} \circ \mathrm{d}f^{-1}$$

and such that

$$\frac{\partial}{\partial t} = \mathrm{d}f(T) + \nu N.$$

Moreover the immersion is unique up to a global isometry of $\mathbb{M}^n \times \mathbb{R}$ preserving the orientations of both $\mathbb{M}^n$ and $\mathbb{R}$.

To prove this theorem, we consider a local orthonormal frame $(e_1, \dots, e_n)$ on $\mathcal{V}$ and the forms ω^i, ω^{n+1}, ω^i_j, ω^{n+1}_j, ω^i_{n+1} and ω^{n+1}_{n+1} as in section 2.2. We set $\mathbb{E}^{n+2} = \mathbb{R}^{n+2}$ or $\mathbb{E}^{n+2} = \mathbb{L}^{n+2}$ (according to $\mathbb{M}^n$). We denote by $(E_0, \dots, E_{n+1})$ the canonical frame of $\mathbb{E}^{n+2}$ (with $\langle E_0, E_o\rangle = -1$ in the case of $\mathbb{L}^{n+2}$); in particular we have $E_{n+1} = \frac{\partial}{\partial t}$. We set

$$T^k = \langle T, e_k\rangle, \quad T^{n+1} = \nu, \quad T^0 = 0.$$

Moreover we set

$$\omega^0_j(e_k) = \kappa(T^jT^k - \delta^k_j), \quad \omega^0_{n+1}(e_k) = \kappa\nu T^k,$$

$$\omega^i_0 = -\kappa\omega^0_i, \quad \omega^{n+1}_0 = -\kappa\omega^0_{n+1}, \quad \omega^0_0 = 0.$$

We define the one-form η on $\mathcal{V}$ by

$$\eta(X) = \langle T, X\rangle.$$

In the frame $(e_1, \dots, e_n)$ we have $\eta = \sum_k T^k \omega^k$. Finaly we define the following matrix of one-forms:

$$\Omega = (\omega^\alpha_\beta) \in \mathcal{M}_{n+2}(\mathbb{R}),$$

the indices going from 0 to $n+1$.

From now on we assume that the hypotheses of theorem 3.3 are satisfied. We first prove some technical lemmas that are consequences of the compatibility equations.

Lemma 3.4. *We have*

$$\mathrm{d}\eta = 0.$$

Proof. We have

$$\begin{aligned} \mathrm{d}\eta(X,Y) &= X\cdot\eta(Y) - Y\cdot\eta(X) - \eta([X,Y]) \\ &= \langle\nabla_X T, Y\rangle - \langle\nabla_Y T, X\rangle \\ &= \langle\nu \mathrm{S}X, Y\rangle - \langle\nu\mathrm{S}Y, X\rangle \\ &= 0. \end{aligned}$$

We have used condition (9). □

Lemma 3.5. *We have*

$$\mathrm{d}T^\alpha = \sum_\gamma T^\gamma\omega^\gamma_\alpha.$$

Proof. This is a consequence of condition (9) for $\alpha = j$, of condition (10) for $\alpha = n+1$, and of the definitions for $\alpha = 0$. □

Lemma 3.6. *We have*

$$\mathrm{d}\Omega + \Omega\wedge\Omega = 0.$$

Proof. We set $\Psi = \mathrm{d}\Omega + \Omega\wedge\Omega$ and $R^i_{klj} = \langle \mathrm{R}(e_k, e_l)e_j, e_i\rangle$.

By proposition 2.4 we have

$$\Psi^i_j = -\frac{1}{2}\sum_k\sum_l R^i_{klj}\omega^k\wedge\omega^l + \omega^i_{n+1}\wedge\omega^{n+1}_j + \omega^i_0\wedge\omega^0_j.$$

Since the Gauss equation (7) is satisfied we have

$$R^i_{klj} = \bar{R}^i_{klj} + \omega^{n+1}_j\wedge\omega^{n+1}_i(e_k, e_l)$$

with

$$\bar{R}^i_{klj} = \kappa(\delta^k_j\delta^l_i - \delta^l_j\delta^k_i - T^lT^i\delta^k_j - T^kT^j\delta^l_i + T^kT^i\delta^l_j + T^lT^j\delta^k_i).$$

On the other hand, a computation shows that $\omega^i_0\wedge\omega^0_j(e_k,e_l) = \bar{R}^i_{klj}$. Thus we have $R^i_{klj} = \omega^i_{n+1}\wedge\omega^{n+1}_j(e_k,e_l) + \omega^i_0\wedge\omega^0_j(e_k,e_l)$. We conclude that $\Psi^i_j = 0$.

By proposition 2.4 we have

$$\Psi^{n+1}_j = \frac{1}{2}\sum_k\sum_l\langle\nabla_{e_k}\mathrm{S}e_l - \nabla_{e_l}\mathrm{S}e_k - \mathrm{S}[e_k,e_l], e_j\rangle\omega^k\wedge\omega^l + \omega^{n+1}_0\wedge\omega^0_j.$$

Since the Codazzi equation (8) is satisfied we have

$$\langle\nabla_{e_k}\mathrm{S}e_l - \nabla_{e_l}\mathrm{S}e_k - \mathrm{S}[e_k,e_l], e_j\rangle = \kappa(T^lT^{n+1}\delta^k_j - T^kT^{n+1}\delta^l_j).$$

On the other hand, a computation shows that $\omega^{n+1}_0\wedge\omega^0_j(e_k,e_l) = \kappa(T^kT^{n+1}\delta^l_j - T^lT^{n+1}\delta^k_j)$. We conclude that $\Psi^{n+1}_j = 0$.

We have $\omega_j^0 = \kappa(T^j\eta - \omega^j)$. Since $\mathrm{d}\eta = 0$ (by lemma 3.4) we get

$$\mathrm{d}\omega_j^0 = \kappa(\mathrm{d}T^j \wedge \eta - \mathrm{d}\omega^j) = \kappa \mathrm{d}T^j \wedge \eta + \kappa \sum_k \omega_k^j \wedge \omega^k$$

by proposition 2.4. Thus by a straightforward computation we get

$$\begin{aligned}
\Psi_j^0(e_p, e_q) &= \mathrm{d}\omega_j^0(e_p, e_q) + \sum_k \omega_k^0 \wedge \omega_j^k(e_p, e_q) + \omega_{n+1}^0 \wedge \omega_j^{n+1}(e_p, e_q) \\
&= \kappa(\mathrm{d}T^j(e_p)\eta(e_q) - \mathrm{d}T^j(e_q)\eta(e_p) + \omega_q^j(e_p) - \omega_p^j(e_q)) \\
&\quad +\kappa\left(T^p \sum_k T^k\omega_j^k(e_q) - T^q \sum_k T^k \omega_j^k(e_p) - \omega_j^p(e_q) + \omega_j^q(e_p)\right) \\
&\quad +\kappa\left(T^pT^{n+1}\omega_j^{n+1}(e_q) - T^qT^{n+1}\omega_j^{n+1}(e_p)\right).
\end{aligned}$$

Using the definition of η and lemma 3.5 for $\alpha = j$,we conclude that $\hat{\Psi}_j^{n+2} = 0$.

We have $\omega_{n+1}^0 = \kappa T^{n+1}\eta$, and so $\mathrm{d}\omega_{n+1}^0 = \kappa \mathrm{d}T^{n+1} \wedge \eta$ by lemma 3.4. Thus by a straightforward computation we get

$$\begin{aligned}
\Psi_{n+1}^0(e_p, e_q) &= \mathrm{d}\omega_{n+1}^0(e_p, e_q) + \sum_k \omega_k^0 \wedge \omega_{n+1}^k(e_p, e_q) \\
&= \kappa(T^q\mathrm{d}T^{n+1}(e_p) - T^p\mathrm{d}T^{n+1}(e_q)) \\
&\quad +\kappa\left(T^p \sum_k T^k\omega_{n+1}^k(e_q) - T^q \sum_k T^k\omega_{n+1}^k(e_p)\right) \\
&\quad +\kappa(-\omega_{n+1}^p(e_q) + \omega_{n+1}^q(e_p)).
\end{aligned}$$

The last two terms cancel because S is symmetric. Using lemma 3.5 for $\alpha = n+1$, we conclude that $\Psi_{n+1}^0 = 0$.

The fact that $\Psi_0^0 = 0$ and $\Psi_{n+1}^{n+1} = 0$ is clear. We conclude by noticing that $\Psi_{n+1}^i = -\Psi_i^{n+1} = 0$. □

For $y \in \mathcal{V}$, let $\mathcal{Z}(y)$ be the set of matrices $Z \in \mathrm{SO}^+(\mathbb{E}^{n+2})$ such that the coefficients of the last line of Z are the $T^\beta(y)$. It is a manifold of dimension $\frac{n(n+1)}{2}$ (since the map $F : \mathrm{SO}^+(\mathbb{E}^{n+2}) \to \mathbb{S}(\mathbb{E}^{n+2}), Z \mapsto (Z_\beta^{n+1})_\beta$ (i.e., $F(Z)$ is the last line of Z), where $\mathbb{S}(\mathbb{E}^{n+2}) = \{x \in \mathbb{E}^{n+2}; \langle E, E\rangle = 1\}$, is a submersion).

We now prove the following proposition.

Proposition 3.7. *Assume that the compatibility equations for $\mathbb{M}^n \times \mathbb{R}$ are satisfied. Let $y_0 \in \mathcal{V}$ and $A_0 \in \mathcal{Z}(y_0)$. Then there exist a neighbourhood U_1 of y_0 in $\mathcal{V}$ and a unique map $A : U_1 \to \mathrm{SO}^+(\mathbb{E}^{n+2})$ such that*

$$A^{-1}\mathrm{d}A = \Omega,$$

$$\forall y \in U_1, \quad A(y) \in \mathcal{Z}(y),$$

$$A(y_0) = A_0.$$

Proof. Let U be a coordinate neighbourhood in $\mathcal{V}$. The set

$$\mathcal{F} = \{(y, Z) \in U \times \mathrm{SO}^+(\mathbb{E}^{n+2}); Z \in \mathcal{Z}(y)\}$$

is a manifold of dimension $n + \frac{n(n+1)}{2}$, and

$$\mathrm{T}_{(y,Z)}\mathcal{F} = \{(u, \zeta) \in \mathrm{T}_yU \oplus \mathrm{T}_Z\mathrm{SO}^+(\mathbb{E}^{n+2}); \zeta_\beta^{n+1} = (\mathrm{d}T^\beta)_y(u)\}.$$

Indeed, in the neighbourhood of point of U there exists a map $y \mapsto M(y) \in \mathrm{SO}^+(\mathbb{E}^{n+2})$ such that the last line of $M(y)$ is $(T^\beta(y))_\beta$, and we have $Z \in \mathcal{Z}(y)$ if and only if

$$ZM(y)^{-1} = \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}$$

for some $B \in \mathrm{SO}^+(\mathbb{E}^{n+1})$; then, if φ is a local parametrization of the set of such matrices, the map $(y, v) \mapsto (y, \varphi(v)M(y))$ is a local parametrization of $\mathcal{F}$.

Let Z denote the projection $U \times \mathrm{SO}^+(\mathbb{E}^{n+2}) \to \mathrm{SO}^+(\mathbb{E}^{n+2}) \subset \mathcal{M}_{n+2}(\mathbb{R})$. We consider on $\mathcal{F}$ the following matrix of 1-forms:

$$\Theta = Z^{-1}\mathrm{d}Z - \Omega,$$

namely for $(y, Z) \in \mathcal{F}$ we have

$$\Theta_{(y,Z)} : \mathrm{T}_{(y,Z)}\mathcal{F} \to \mathcal{M}_{n+2}(\mathbb{R}),$$
$$\Theta_{(y,Z)}(u, \zeta) = Z^{-1}\zeta - \Omega_y(u).$$

We claim that, for each $(y, Z) \in \mathcal{F}$, the space

$$\mathcal{D}(y, Z) = \ker \Theta_{(y,Z)}$$

has dimension n. We first notice that the matrix Θ belongs to $\mathfrak{so}(\mathbb{E}^{n+2})$ since Ω and $Z^{-1}\mathrm{d}Z$ do. Moreover we have

$$(Z\Theta)^{n+1}_\beta = \mathrm{d}Z^{n+1}_\beta - \sum_\gamma Z^{n+1}_\gamma \omega^\gamma_\beta = \mathrm{d}T^\beta - \sum_\gamma T^\gamma \omega^\gamma_\beta = 0$$

by lemma 3.5. Thus the values of $\Theta_{(y,Z)}$ lie in the space

$$\mathcal{H} = \{H \in \mathfrak{so}^+(\mathbb{E}^{n+2}); (ZH)^{n+1}_\beta = 0\},$$

which has dimension $\frac{n(n+1)}{2}$ (indeed, the map $F : \mathrm{SO}^+(\mathbb{E}^{n+2}) \to \mathbb{S}(\mathbb{E}^{n+2}), Z \mapsto (Z^{n+1}_\beta)_\beta$ is a submersion, and we have $H \in \mathcal{H}$ if and only if $ZH \in \ker(\mathrm{d}F)_Z$). Moreover, the space $\mathrm{T}_{(y,Z)}\mathcal{F}$ contains the subspace $\{(0, ZH); H \in \mathcal{H}\}$, and the restriction of $\Theta_{(y,Z)}$ on this subspace is the map $(0, ZH) \mapsto H$. Thus $\Theta_{(y,Z)}$ is onto $\mathcal{H}$, and consequently the linear map $\Theta_{(y,Z)}$ has rank $\frac{n(n+1)}{2}$. This finishes proving the claim.

We now prove that the distribution $\mathcal{D}$ is involutive. Using lemma 3.6 we get

$$\begin{aligned} \mathrm{d}\Theta &= -Z^{-1}\mathrm{d}Z \wedge Z^{-1}\mathrm{d}Z - \mathrm{d}\Omega \\ &= -(\Theta + \Omega) \wedge (\Theta + \Omega) - \mathrm{d}\Omega \\ &= -\Theta \wedge \Theta - \Theta \wedge \Omega - \Omega \wedge \Theta. \end{aligned}$$

From this formula we deduce that if $\xi_1, \xi_2 \in \mathcal{D}$, then $\mathrm{d}\Theta(\xi_1, \xi_2) = 0$, and so $\Theta([\xi_1, \xi_2]) = \xi_1 \cdot \Theta(\xi_2) - \xi_2 \cdot \Theta(\xi_1) - \mathrm{d}\Theta(\xi_1, \xi_2) = 0$, i.e., $[\xi_1, \xi_2] \in \mathcal{D}$. Thus the distribution $\mathcal{D}$ is involutive, and so, by the theorem of Frobenius, it is integrable.

Let $\mathcal{A}$ be the integral manifold through (y_0, A_0). If $\zeta \in \mathrm{T}_{A_0}\mathrm{SO}^+(\mathbb{E}^{n+2})$ is such that $(0, \zeta) \in \mathrm{T}_{(y_0,A_0)}\mathcal{A} = \mathcal{D}(y_0, A_0)$, then we have $0 = \Theta_{(y_0,A_0)}(0, \zeta) = A_0^{-1}\zeta$. This proves that

$$\mathrm{T}_{(y_0,A_0)}\mathcal{A} \cap \left(\{0\} \times \mathrm{T}_{A_0}\mathrm{SO}^+(\mathbb{E}^{n+2})\right) = \{0\}.$$

Thus the manifold $\mathcal{A}$ is locally the graph of a function $A : U_1 \to \mathrm{SO}^+(\mathbb{E}^{n+2})$ where U_1 is a neighbourhood of y_0 in U. By construction, this map satisfies the properties of proposition 3.7 and is unique. □

We now prove the theorem.

Proof of theorem 3.3. Let $y_0 \in \mathcal{V}$, $A \in \mathcal{Z}(y_0)$ and $t_0 \in \mathbb{R}$. We consider on $\mathcal{V}$ a local orthonormal frame $(e_1, \dots, e_n)$ in the neighbourhood of y_0 and we keep the same notations. Then by proposition 3.7 there exists a unique map $A : U_1 \to \mathrm{SO}^+(\mathbb{E}^{n+2})$ such that

$$A^{-1}\mathrm{d}A = \Omega,$$
$$\forall y \in U_1, \quad A(y) \in \mathcal{Z}(y),$$
$$A(y_0) = A_0,$$

where U_1 is a neighbourhood of y_0, which we can assume simply connected.

We set $f^0 = A_0^0$, $f^i = A_0^i$ and we call f^{n+1} the unique function on U_1 such that $\mathrm{d}f^{n+1} = \eta$ and $f^{n+1}(y_0) = t_0$ (this function exists since U_1 is simply connected and $\mathrm{d}\eta = 0$). Thus we defined a map $f : U_1 \to \mathbb{E}^{n+2}$. Since $A_0^{n+1} = T^0 = 0$ and $A \in \mathrm{SO}^+(\mathbb{E}^{n+2})$, in the case of $\mathbb{S}^n \times \mathbb{R}$ we have $(f^0)^2 + \sum_i (f^i)^2 = \sum_\alpha (A_0^\alpha)^2 = 1$, and in the case of $\mathbb{H}^n \times \mathbb{R}$ we have $-(f^0)^2 + \sum_i (f^i)^2 = -(A_0^0)^2 + \sum_i (A_0^i)^2 + (A_0^{n+1})^2 = -1$ and $f^0 = A_0^0 > 0$. Thus in both cases we have $(f^0, \dots, f^n) \in \mathbb{M}^n$, i.e., the values of f lie in $\mathbb{M}^n \times \mathbb{R}$.

Since $\mathrm{d}A = A\Omega$, we have, for $\alpha < n+1$,

$$\begin{aligned} \mathrm{d}f^\alpha(e_k) &= \sum_j A_j^\alpha \omega_0^j(e_k) + A_{n+1}^\alpha \omega_0^{n+1}(e_k) \\ &= \sum_j A_j^\alpha (\delta_j^k - T^j T^k) - A_{n+1}^\alpha T^{n+1} T^k \\ &= A_k^\alpha - T^k \sum_\beta A_\beta^\alpha A_\beta^{n+1} \\ &= A_k^\alpha, \end{aligned}$$

and

$$\mathrm{d}f^{n+1}(e_k) = \eta(e_k) = T^k = A_k^{n+1}.$$

This means that $\mathrm{d}f(e_k)$ is given by the colunm k of the matrix A.

Since A is an invertible matrix, $\mathrm{d}f$ has rank n, and so f is an immersion. And since $A \in \mathrm{SO}^+(\mathbb{E}^{n+2})$, we have $\langle \mathrm{d}f(e_p), \mathrm{d}f(e_q) \rangle = \delta_q^p$, and so f is an isometry.

The columns of $A(y)$ form a direct orthonormal frame of $\mathbb{E}^{n+2}$. Columns 1 to n form a direct orthonormal frame of $\mathrm{T}_{f(y)} f(\mathcal{V})$ and column 0 is the projection of $f(y)$ on $\mathbb{M}^n \times \{0\}$, i.e., the unit normal $\bar{N}(f(y))$ to $\mathbb{M}^n \times \mathbb{R}$ at the point $f(y)$. Thus column $(n+1)$ is the unit normal $N(f(y))$ to $f(\mathcal{V})$ in $\mathbb{M}^n \times \mathbb{R}$ at the point $f(y)$.

We set $X_j = \mathrm{d}f(e_j)$. Then we have

$$\begin{aligned} \langle \mathrm{d}X_j(X_k), N \rangle &= \sum_\alpha \mathrm{d}A_j^\alpha(e_k) A_{n+1}^\alpha = \sum_\alpha \sum_\gamma A_\gamma^\alpha A_{n+1}^\alpha \omega_j^\gamma(e_k) \\ &= \omega_j^{n+1}(e_k) = \langle \mathrm{S}e_k, e_j \rangle. \end{aligned}$$

This means that the shape operator of $f(\mathcal{V})$ in $\mathbb{M}^n \times \mathbb{R}$ is $\mathrm{d}f \circ \mathrm{S} \circ \mathrm{d}f^{-1}$.

Finally, the coefficients of the vertical vector $\frac{\partial}{\partial t} = E_{n+1}$ in the orthonormal frame $(\bar{N}, X_1, \dots, X_n, N)$ are given by the last line of A. Since $A(y) \in \mathcal{Z}(y)$ for all $y \in U_2$ we get

$$\frac{\partial}{\partial t} = \sum_j T^j X_j + T^{n+1} N = \mathrm{d}f(T) + \nu N.$$

We now prove that the local immersion is unique up to a global isometry of $\mathbb{M}^n \times \mathbb{R}$. Let $\tilde{f} : U_3 \to \mathbb{M}^n \times \mathbb{R}$ be another immersion satisfying the conclusion

of the theorem, where U_3 is a simply connected neighbourhood of y_0 included in U_1, let $(\tilde{X}_\beta)$ be the associated frame (i.e., $\tilde{X}_j = \mathrm{d}\tilde{f}(e_j)$, $\tilde{X}_{n+1}$ is the normal of $\tilde{f}(\mathcal{V})$ in $\mathbb{M}^n \times \mathbb{R}$ and $\tilde{X}_0$ is the normal to $\mathbb{M}^n \times \mathbb{R}$ in $\mathbb{E}^{n+2}$) and let $\tilde{A}$ the matrix of the coordinates of the frame $(\tilde{X}_\beta)$ in the frame (E_α). Up to a direct isometry of $\mathbb{M}^n \times \mathbb{R}$, we can assume that $f(y_0) = \tilde{f}(y_0)$ and that the frames $(X_\beta(y_0))$ and $(\tilde{X}_\beta(y_0))$ coincide, i.e., $A(y_0) = \tilde{A}(y_0)$. We notice that this isometry necessarily fixes $\frac{\partial}{\partial t}$ since the T^α are the same for x and $\tilde{x}$. The matrices A and $\tilde{A}$ satisfy $A^{-1}\mathrm{d}A = \Omega$ and $\tilde{A}^{-1}\mathrm{d}\tilde{A} = \Omega$ (see section 2.3), $A(y), \tilde{A}(y) \in \mathcal{Z}(y)$ and $A(y_0) = \tilde{A}(y_0)$, thus by the uniqueness of the solution of the equation in proposition 3.7 we get $A(y) = \tilde{A}(y)$. Considering the columns 0 of these matrices, we get $f^i = \tilde{f}^i$ and $f^0 = \tilde{f}^0$. Finally we have $\mathrm{d}f^{n+1} = \eta = \mathrm{d}\tilde{f}^{n+1}$ and $f^{n+1}(y_0) = \tilde{f}^{n+1}(y_0)$, thus we have $f^{n+1} = \tilde{f}^{n+1}$. This finishes proving that $f = \tilde{f}$ on U_3.

Finally we prove that this local immersion f can be extended to $\mathcal{V}$ in a unique way. Let $y_1 \in \mathcal{V}$. Then there exists a curve $\Gamma : [0,1] \to \mathcal{V}$ such that $\Gamma(0) = y_0$ and $\Gamma(1) = y_1$. Each point of Γ has a neighbourhood such that there exists an isometric immersion (unique up to an isometry of $\mathbb{M}^n \times \mathbb{R}$ preserving the orientations of $\mathbb{M}^n$ and $\mathbb{R}$) of this neighbourhood satisfying the properties of the theorem. From this family of neighbourhoods we can extract a finite family $(W_1, \dots, W_p)$ covering Γ with $W_1 = U_1$. Then the above uniqueness argument shows that we can extend successively the immersion f to the W_k in a unique way. In particular $f(y_1)$ is defined. Moreover, this value $f(y_1)$ does not depend on the choice of the curve Γ joining y_0 to y_1 because $\mathcal{V}$ is simply connected. □

Proposition 3.8. *If* $(\mathrm{d}s^2, \mathrm{S}, T, \nu)$ *satisfies the compatibilty equations and correspond to an immerion* $f : \Sigma \to \mathbb{M}^n \times \mathbb{R}$, *then* $(\mathrm{d}s^2, -\mathrm{S}, T, -\nu)$, $(\mathrm{d}s^2, -\mathrm{S}, -T, \nu)$ *and* $(\mathrm{d}s^2, \mathrm{S}, -T, -\nu)$ *also satisfy the compatibilty equations and they correspond to the immersion* $\sigma \circ f$ *where* σ *is an isometry of* $\mathbb{M}^n \times \mathbb{R}$

(1) *reversing the orientation of* $\mathbb{M}^n$ *and preserving the orientation of* $\mathbb{R}$ *in the case of* $(\mathrm{d}s^2, -\mathrm{S}, T, -\nu)$,
(2) *preserving the orientation of* $\mathbb{M}^n$ *and reversing the orientation of* $\mathbb{R}$ *in the case of* $(\mathrm{d}s^2, -\mathrm{S}, -T, \nu)$,
(3) *reversing the orientations of both* $\mathbb{M}^n$ *and* $\mathbb{R}$ *in the case of* $(\mathrm{d}s^2, \mathrm{S}, -T, -\nu)$.

Proof. We deal with the first case (the two others are similar). Let $\hat{f} = \sigma \circ f$. Then the normal to $\mathbb{M}^n \times \mathbb{R}$ is $\sigma \circ \bar{N}$, and since σ reverses the orientation of $\mathbb{M}^n \times \mathbb{R}$ the normal to $\hat{f}(\mathcal{V})$ in $\mathbb{M}^n \times \mathbb{R}$ is $\hat{N} = -\sigma \circ N$. From this we deduce that $\hat{\mathrm{S}} = -\mathrm{S}$. Moreover we have $\frac{\partial}{\partial t} = \mathrm{d}f(T) + \nu N$, and so, since σ preserves the orientation of $\mathbb{R}$ we have
$$\frac{\partial}{\partial t} = \sigma \circ \mathrm{d}f(T) + \nu\sigma \circ N = \mathrm{d}\hat{f}(T) - \nu\hat{N}.$$
We conclude that $\hat{T} = T$ and $\hat{\nu} = -\nu$. □

3.3. **Remark: another proof in the case of** $\mathbb{H}^n \times \mathbb{R}$**.** In this section we outline another proof of theorem 3.3 in the case of $\mathbb{H}^n \times \mathbb{R}$ that does not involve the Lorentz space. Greek letters will denote indices between 1 and $n+1$.

We first consider an orientable hypersurface $\mathcal{V}$ of an $(n+1)$-dimensionnal Riemannian manifold $\bar{\mathcal{V}}$. Let $(e_1, \dots, e_n)$ be a local orthonormal frame on $\mathcal{V}$, e_{n+1} the normal to $\mathcal{V}$, and $(E_1, \dots, E_{n+1})$ a local orthonormal frame on $\bar{\mathcal{V}}$. We denote by ∇ and $\bar{\nabla}$ the Riemannian connections on $\mathcal{V}$ and $\bar{\mathcal{V}}$ respectively, and by S the shape

operator of $\mathcal{V}$ (with respect to the normal e_{n+1}). We define the forms ω^α, ω^α_β on $\mathcal{V}$ as in section 2.2. Then we have

$$\bar{\nabla}_{e_k} e_\beta = \sum_\gamma \omega^\gamma_\beta(e_k) e_\gamma.$$

Let $A \in \mathrm{SO}_{n+1}(\mathbb{R})$ be the matrix whose columns are the coordinates of the e_β in the frame (E_α), namely $A^\alpha_\beta = \langle e_\beta, E_\alpha \rangle$. Let $\Omega = (\omega^\alpha_\beta) \in \mathcal{M}_{n+1}(\mathbb{R})$. The matrix A satisfies the following equation:

$$A^{-1}\mathrm{d}A = \Omega + L(A)$$

with

$$L(A)^\alpha_\beta = \sum_k \left(\sum_{\gamma,\delta,\varepsilon} A^\varepsilon_\alpha A^\gamma_k A^\delta_\beta \bar{\Gamma}^\delta_{\gamma\alpha} \right) \omega^k,$$

where the $\bar{\Gamma}^\delta_{\gamma\alpha}$ are the Christoffel symbols of the frame (E_α). Notice that these matrices have size $n+1$, whereas those of section 2.3 have size $n+2$.

We now assume that $\bar{\mathcal{V}} = \mathbb{H}^n \times \mathbb{R}$ and that $\mathcal{V}$ is a Riemannian manifold of dimension n endowed with S, T, ν satisfying the compatibility equations for $\mathbb{H}^n \times \mathbb{R}$. We consider a local orthonormal frame $(e_1, \dots, e_n)$ on $U \subset \mathcal{V}$, the associated one-forms ω^α, ω^α_β and the matrix of one-forms $\Omega \in \mathcal{M}_{n+1}(\mathbb{R})$.

We use the fact that there exists an orthonormal frame on $\mathbb{H}^n$ whose Christoffel symbols are constant; more precisely, we can choose the frame (E_α) on $\mathbb{H}^n \times \mathbb{R}$ such that $\bar{\Gamma}^i_{ij} = -\bar{\Gamma}^j_{ii} = \frac{1}{\sqrt{n}}$ for $i \neq j$, $i, j \leqslant n$ and all the other Christoffel symbols vanish.

The first step is to prove the following proposition, which is analogous to proposition 3.7.

Proposition 3.9. *Let $y_0 \in \mathcal{V}$ and $A_0 \in \mathcal{Z}(y_0)$. Then there exist a neighbourhood U_1 of y_0 in $\mathcal{V}$ and a unique map $A : U_1 \to \mathrm{SO}_{n+1}(\mathbb{R})$ such that*

$$A^{-1}\mathrm{d}A = \Omega + L(A),$$

$$\forall y \in U_1, \quad A(y) \in \mathcal{Z}(y),$$

$$A(y_0) = A_0,$$

where $\mathcal{Z}(y)$ is defined in a way analogous to that of section 3.2.

To prove this proposition, we introduce the form $\Theta = Z^{-1}\mathrm{d}Z - \Omega - L(Z)$ on $\mathcal{F} = \{(y, Z) \in U \times \mathrm{SO}_{n+1}(\mathbb{R}); Z \in \mathcal{Z}(y)\}$; this is well defined since the Christoffel symbols are constant. A long calculation shows that the distribution $\mathcal{D}(y, Z) = \ker \Theta_{(y,Z)}$ is involutive. We conclude as in the proof of proposition 3.7.

The second step is to prove the following proposition.

Proposition 3.10. *Let $x_0 \in \mathbb{H}^n \times \mathbb{R}$. There exist a neighbourhood U_2 of y_0 contained in U_1 and a function $f : U_2 \to \mathbb{H}^n \times \mathbb{R}$ such that*

$$\mathrm{d}f = (B \circ f)A\omega,$$

$$f(y_0) = x_0,$$

where ω is the column $(\omega^1, \dots, \omega^n, 0)$ and, for $x \in \mathbb{H}^n \times \mathbb{R}$, $B(x) \in \mathcal{M}_{n+1}(\mathbb{R})$ is the matrix of the coordinates of the frame $(E_\alpha(x))$ in the frame $(\frac{\partial}{\partial x^\alpha})$ (we choose the upper half-space model for $\mathbb{H}^n$).

To prove it, we consider the form $B^{-1}\mathrm{d}x - A\omega$ on $U_1 \times \bar{\mathcal{V}}$, and we show that its kernel again defines an involutive distribution.

The last step is to check that this map f satisfies the conclusions of theorem 3.3.

4. Applications to minimal surfaces in $\mathbb{M}^2 \times \mathbb{R}$

4.1. **The associate family.** Let $\mathbb{M}^2 = \mathbb{S}^2$ or $\mathbb{M}^2 = \mathbb{H}^2$. Let Σ be a Riemann surface with a metric $\mathrm{d}s^2$ (which we also denote by $\langle \cdot, \cdot \rangle$), ∇ its Riemannian connection, and J the rotation of angle $\frac{\pi}{2}$ on $\mathrm{T}\Sigma$. Let S be a field of symmetric operators $\mathrm{S}_y : \mathrm{T}_y\Sigma \to \mathrm{T}_y\Sigma$. Let T be a vector field on Σ and ν a smooth function on Σ such that $||T||^2 + \nu^2 = 1$.

Proposition 4.1. *Assume that* S *is trace-free and that* $(\mathrm{d}s^2, \mathrm{S}, T, \nu)$ *satisfies the compatibility equations for* $\mathbb{M}^2 \times \mathbb{R}$*. For* $\theta \in \mathbb{R}$ *we set*

$$\mathrm{S}_\theta = e^{\theta \mathrm{J}}\mathrm{S} = (\cos\theta)\mathrm{S} + (\sin\theta)\mathrm{JS},$$

$$T_\theta = e^{\theta \mathrm{J}}T = (\cos\theta)T + (\sin\theta)\mathrm{J}T,$$

i.e., S_θ *and* T_θ *are obtained by rotating* S *and* T *by the angle* θ*.*

Then S_θ *is symmetric and trace-free,* $||T_\theta||^2 + \nu^2 = 1$ *and* $(\mathrm{d}s^2, \mathrm{S}_\theta, T_\theta, \nu)$ *satisfies the compatibility equations for* $\mathbb{M}^2 \times \mathbb{R}$*.*

Proof. The fact that S_θ is symmetric and trace-free comes from an elementary computation. Moreover we have $||T_\theta|| = ||T||$. We notice that, since $\dim \Sigma = 2$, the Gauss equation (7) is equivalent to

$$K = \det \mathrm{S} + \kappa(1 - ||T||^2)$$

where K is the Gauss curvature of $\mathrm{d}s^2$. Since $\det(e^{\theta \mathrm{J}}) = 1$, we have $\det \mathrm{S}_\theta = \det \mathrm{S}$, and so the Gauss equation is satisfied for $(\mathrm{d}s^2, \mathrm{S}_\theta, T_\theta, \nu)$.

Since $e^{\theta \mathrm{J}}$ commutes with ∇_X (see [AR03], section 3.2) and preserves the metric, equations (9) and (10) are also satisfied for $(\mathrm{d}s^2, \mathrm{S}_\theta, T_\theta, \nu)$.

To prove that the Codazzi equation (8) is satisfied by $(\mathrm{d}s^2, \mathrm{S}_\theta, T_\theta, \nu)$, we first notice that, since $\nabla_X e^{\theta \mathrm{J}}\mathrm{S}Y - \nabla_Y e^{\theta \mathrm{J}}\mathrm{S}X - e^{\theta \mathrm{J}}\mathrm{S}[X, Y] = e^{\theta \mathrm{J}}(\nabla_X \mathrm{S}Y - \nabla_Y \mathrm{S}X - \mathrm{S}[X, Y])$, it suffices to prove that

$$\langle e^{\theta \mathrm{J}}T, Y\rangle X - \langle e^{\theta \mathrm{J}}T, X\rangle Y = e^{\theta \mathrm{J}}(\langle T, Y\rangle X - \langle T, X\rangle Y).$$

This is obvious at a point where $X = 0$. At a point where $X \neq 0$, we can write $Y = \lambda X + \mu \mathrm{J}X$, and a computation shows that both expressions are equal to $\mu\cos\theta\langle T, \mathrm{J}X\rangle X + \mu\sin\theta\langle T, X\rangle X - \mu\cos\theta\langle T, X\rangle \mathrm{J}X + \mu\sin\theta\langle T, \mathrm{J}X\rangle \mathrm{J}X$. □

Theorem 4.2. *Let* Σ *be a simply connected Riemann surface and* $x : \Sigma \to \mathbb{M}^2 \times \mathbb{R}$ *a conformal minimal immersion. Let* N *be the induced normal. Let* S *be the symmetric operator on* Σ *induced by the shape operator of* $x(\Sigma)$*. Let* T *be the vector field on* Σ *such that* $\mathrm{d}x(T)$ *is the projection of* $\frac{\partial}{\partial t}$ *onto* $\mathrm{T}(x(\Sigma))$*. Let* $\nu = \langle N, \frac{\partial}{\partial t}\rangle$*.*

Let $z_0 \in \Sigma$*. Then there exists a unique family* $(x_\theta)_{\theta \in \mathbb{R}}$ *of conformal minimal immersions* $x_\theta : \Sigma \to \mathbb{M}^2 \times \mathbb{R}$ *such that:*

1. $x_\theta(z_0) = x(z_0)$ *and* $(\mathrm{d}x_\theta)_{z_0} = (\mathrm{d}x)_{z_0}$*,*
2. *the metrics induced on* Σ *by* x *and* x_θ *are the same,*
3. *the symmetric operator on* Σ *induced by the shape operator of* $x_\theta(\Sigma)$ *is* $e^{\theta \mathrm{J}}\mathrm{S}$*,*
4. $\frac{\partial}{\partial t} = \mathrm{d}x_\theta(e^{\theta \mathrm{J}}T) + \nu N_\theta$ *where* N_θ *is the unit normal to* x_θ*.*

Moreover we have $x_0 = x$ and the family (x_θ) is continuous with respect to θ.

The family of immersions $(x_\theta)_{\theta\in\mathbb{R}}$ is called the associate family of the immersion x. The immersion $x_{\frac{\pi}{2}}$ is called the conjugate immersion of the immersion x. The immersion x_π is called the opposite immersion of the immersion x.

Proof. Let $\mathrm{d}s^2$ be the metric on Σ induced by x. Then $(\mathrm{d}s^2, \mathrm{S}, T, \nu)$ satisfies the compatibility equations for $\mathbb{M}^2 \times \mathbb{R}$. Thus, by proposition 4.1, $(\mathrm{d}s^2, e^{\theta \mathrm{J}}\mathrm{S}, e^{\theta \mathrm{J}}T, \nu)$ also does. Thus by theorem 3.3 there exists a unique immersion x_θ satisfying the properties of the theorem. The fact that $x_0 = x$ is clear.

Finally, $(\mathrm{d}s^2, e^{\theta \mathrm{J}}\mathrm{S}, e^{\theta \mathrm{J}}T, \nu)$ defines a matrix of one-forms Ω_θ and a matrix of functions A_θ satisfying $A_\theta^{-1}\mathrm{d}A_\theta = \Omega_\theta$ (by proposition 3.7). By continuity of Ω_θ with respect to θ we obtain the continuity of A_θ with respect to θ, and then the continuity of x_θ with respect to θ. $\square$

Remark 4.3. Let $\tau : \Sigma' \to \Sigma$ be a conformal diffeomorphism. If τ preserves the orientation, then $(x\circ\tau)_\theta = x_\theta\circ\tau$; if τ reverses the orientation, then $(x\circ\tau)_\theta = x_{-\theta}\circ\tau$.

In the sequel, we will speak of associate and conjugate immersions even if condition 1 is not satisfied, i.e., we will consider these notions up to isometries of $\mathbb{M}^2 \times \mathbb{R}$ preserving the orientations of both $\mathbb{M}^2$ and $\mathbb{R}$.

Remark 4.4. The opposite immersion is $x_\pi = \sigma \circ x$ where σ is an isometry of $\mathbb{M}^2 \times \mathbb{R}$ preserving the orientation of $\mathbb{M}^2$ and reversing the orientation of $\mathbb{R}$ (see proposition 3.8, case 2).

Remark 4.5. This associate family for minimal immersions in $\mathbb{M}^2 \times \mathbb{R}$ is analogous to the associate family for minimal immersions in $\mathbb{R}^3$. Conformal minimal immersions in $\mathbb{R}^3$ are given by the Weierstrass representation:

$$x(z) = x(z_0) + \mathrm{Re}\int_{z_0}^{z} (1 - g^2, i(1 + g^2), 2g)\omega$$

where g is a meromorphic function on Σ (the Gauss map) and ω a holomorphic one-form. Then the associate immersions are

$$x_\theta(z) = x(z_0) + \mathrm{Re}\int_{z_0}^{z} (1 - g^2, i(1 + g^2), 2g)e^{-i\theta}\omega.$$

Let $x = (\varphi, h) : \Sigma \to \mathbb{M}^2 \times \mathbb{R}$ be a conformal minimal immersion. Then h is a real harmonic function and φ is a harmonic map to $\mathbb{M}^2$. We set

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right).$$

The Hopf differential of φ is the following 2-form (see [Ros02b]):

$$\mathrm{Q}\varphi = 4\left\langle \frac{\partial\varphi}{\partial z}, \frac{\partial\varphi}{\partial z}\right\rangle \mathrm{d}z^2 = \left(\left|\left|\frac{\partial\varphi}{\partial u}\right|\right|^2 - \left|\left|\frac{\partial\varphi}{\partial v}\right|\right|^2 - 2i\left\langle\frac{\partial\varphi}{\partial u}, \frac{\partial\varphi}{\partial v}\right\rangle\right)\mathrm{d}z^2.$$

It is a holomorphic 2-form on Σ, and since x is conformal we have

$$\mathrm{Q}\varphi = -4\left(\frac{\partial h}{\partial z}\right)^2 \mathrm{d}z^2 = -(\mathrm{d}(h + ih^*))^2 = -4\left\langle T, \frac{\partial x}{\partial z}\right\rangle \mathrm{d}z^2$$

where h^* is the harmonic conjugate function of h (i.e., $\frac{\partial h^*}{\partial u} = -\frac{\partial h}{\partial v}$ and $\frac{\partial h^*}{\partial v} = \frac{\partial h}{\partial u}$). The reader can refer to [SY97] for harmonic maps.

Proposition 4.6. *Let $x = (\varphi, h) : \Sigma \to \mathbb{M}^2 \times \mathbb{R}$ be a conformal minimal immersion, and $(x_\theta) = ((\varphi_\theta, h_\theta))$ its associate family of conformal minimal immersions. Let h^* be the harmonic conjugate of h. Then we have*

$$h_\theta = (\cos\theta)h + (\sin\theta)h^*, \quad \mathrm{Q}\varphi_\theta = e^{-2i\theta}\mathrm{Q}\varphi.$$

Proof. We have

$$\begin{aligned}
\frac{\partial h_\theta}{\partial u} = \left\langle \frac{\partial x_\theta}{\partial u}, \frac{\partial}{\partial t} \right\rangle = \left\langle \frac{\partial}{\partial u}, T_\theta \right\rangle &= \cos\theta \left\langle \frac{\partial}{\partial u}, T \right\rangle + \sin\theta \left\langle \frac{\partial}{\partial u}, \mathrm{J}T \right\rangle \\
&= \cos\theta \left\langle \frac{\partial}{\partial u}, T \right\rangle - \sin\theta \left\langle \frac{\partial}{\partial v}, T \right\rangle \\
&= \cos\theta \frac{\partial h}{\partial u} - \sin\theta \frac{\partial h}{\partial v}.
\end{aligned}$$

In the same way we have $\frac{\partial h_\theta}{\partial v} = \cos\theta \left\langle \frac{\partial}{\partial v}, T \right\rangle + \sin\theta \left\langle \frac{\partial}{\partial v}, \mathrm{J}T \right\rangle = \cos\theta \frac{\partial h}{\partial v} + \sin\theta \frac{\partial h}{\partial u}$. This proves that $h_\theta = (\cos\theta)h + (\sin\theta)h^*$. The expression of $\mathrm{Q}\varphi_\theta$ follows immediately. □

Remark 4.7. Recently, Hauswirth, Sá Earp and Toubiana ([HSET04]) defined the following notion of associated immersions in $\mathbb{H}^2 \times \mathbb{R}$: two isometric conformal minimal immersions in $\mathbb{H}^2 \times \mathbb{R}$ are said to be associated if their Hopf differential differ by the multiplication by some constant $e^{i\theta}$. Morover, they proved that two isometric conformal minimal immersions in $\mathbb{H}^2 \times \mathbb{R}$ having the same Hopf differential are equal up to an isometry of $\mathbb{H}^2 \times \mathbb{R}$. Thus the notions of associated immersions in the sense of this paper and in the sense of [HSET04] are equivalent.

In [SET04], Sá Earp and Toubiana ask the following question: if two conformal minimal immersions $x, \tilde{x} : \Sigma \to \mathbb{M}^2 \times \mathbb{R}$ are isometric, then are they associated ? (This result holds for $\mathbb{R}^3$.)

Remark 4.8. Abresch and Rosenberg ([AR03]) defined a holomorphic Hopf differential for constant mean curvature surfaces in $\mathbb{M}^2 \times \mathbb{R}$. For minimal surfaces in $\mathbb{M}^2 \times \mathbb{R}$, this Hopf differential is

$$\begin{aligned}
Q(X,Y) &= -\frac{\kappa}{2}(\langle T, X\rangle\langle T, Y\rangle - \langle T, \mathrm{J}X\rangle\langle T, \mathrm{J}Y\rangle) \\
&\quad + i\frac{\kappa}{2}(\langle T, \mathrm{J}X\rangle\langle T, Y\rangle + \langle T, X\rangle\langle T, \mathrm{J}Y\rangle).
\end{aligned}$$

A computation shows that

$$Q = \frac{\kappa}{2}\mathrm{Q}\varphi.$$

Proposition 4.9. *Let $x : \Sigma \to \mathbb{M}^2 \times \mathbb{R}$ be a conformal minimal immersion. If x does not define a horizontal $\mathbb{M}^2 \times \{t\}$, then the zeros of T are isolated.*

Proof. The height function $h = \langle x, \frac{\partial}{\partial t}\rangle$ satisfies $\mathrm{d}h(X) = \langle T, X\rangle$; thus the zeroes of T are the zeroes of $\mathrm{d}h$. Since h is harmonic, either the zeroes of $\mathrm{d}h$ are isolated or h is constant. The latter case is excluded by hypothesis. □

Remark 4.10. Umbilic points (i.e., zeroes of the shape operator) may be non-isolated: for example helicoids and unduloids in $\mathbb{S}^2 \times \mathbb{R}$ have curves of umbilic points (see section 4.2).

We now give some geometric properties of conjugate surfaces.

The transformation $\mathrm{S} \mapsto \mathrm{JS}$ implies that curvature lines and asymptotic lines are exchanged by conjugation (as in $\mathbb{R}^3$). (More generally the normal curvature and the normal torsion of a curve are swapped up to a sign.) The reader can refer to [Kar01] for geometric properties of conjugate surfaces in $\mathbb{R}^3$.

Moreover, the transformation $T \mapsto \mathrm{J}T$ implies the following transformation: a horizontal curve γ along which the surface is vertical (i.e., $\nu = 0$ along γ and γ' is orthogonal to T) is mapped to vertical curve (i.e., $\nu = 0$ along γ and γ' is proportional to T), and vice versa. We also notice that a minimal surface cannot be horizontal along a horizontal curve unless the minimal surface is a horizontal $\mathbb{M}^2 \times \{t\}$ (indeed, this would imply that $T = 0$ along this curve).

Hence conjugation swaps two pairs of Schwarz reflections:

(1) the symmetry with respect to a vertical plane containing a curvature line becomes the rotation with respect to a horizontal geodesic of $\mathbb{M}^2$, and vice versa,
(2) the symmetry with respect to a horizontal plane containing a curvature line becomes the rotation with respect to a vertical straight line, and vice versa.

The first case is illustrated by a generatrix curve of an unduloid or a catenoid and a horizontal line of a helicoid; the second case is illustrated by the waist circle of an unduloid or a catenoid and the axis of a helicoid. These examples are detailed in sections 4.2 and 4.3.

4.2. **Helicoids and unduloids in $\mathbb{S}^2 \times \mathbb{R}$.** Apart from the horizontal spheres $\mathbb{S}^2 \times \{t\}$ and the vertical cylinders $\mathbb{S}^1 \times \mathbb{R}$ ($\mathbb{S}^1$ being a great circle in $\mathbb{S}^2$), the most simple examples of minimal surfaces in $\mathbb{S}^2 \times \mathbb{R}$ are helicoids and unduloids. Theses surfaces are described in [PR99] and [Ros02b]. They are properly embedded and foliated by circles. Unduloids are rotational and vertically periodic; helicoids are invariant by a screw motion.

Helicoids. For $\beta \neq 0$, the helicoid $\mathcal{H}_\beta$ is given by the following conformal immersion:

$$x(u,v) = \begin{pmatrix} \sin\varphi(u)\cos\beta v \\ \sin\varphi(u)\sin\beta v \\ \cos\varphi(u) \\ v \end{pmatrix},$$

where the function φ satisfies

$$\varphi'(u)^2 = 1 + \beta^2 \sin^2\varphi(u), \quad \varphi''(u) = \beta^2 \sin\varphi(u)\cos\varphi(u). \tag{11}$$

We can assume that $\varphi(0) = 0$ and $\varphi'(u) > 0$. When $\beta > 0$ we say that $\mathcal{H}_\beta$ is a right helicoid; when $\beta < 0$ we say that $\mathcal{H}_\beta$ is a left helicoid.

The normal to $\mathbb{S}^2 \times \mathbb{R}$ in $\mathbb{R}^4$ is

$$\bar{N}(u,v) = \begin{pmatrix} \sin\varphi(u)\cos\beta v \\ \sin\varphi(u)\sin\beta v \\ \cos\varphi(u) \\ 0 \end{pmatrix}.$$

The normal to $\mathcal{H}_\beta$ in $\mathbb{S}^2\times\mathbb{R}$ is

$$N(u,v)=\frac{1}{\varphi'(u)}\begin{pmatrix}\sin\beta v\\ -\cos\beta v\\ 0\\ \beta\sin\varphi(u)\end{pmatrix}.$$

We compute:

$$\left\langle\frac{\partial^2 x}{\partial u^2},N\right\rangle=\left\langle\frac{\partial^2 x}{\partial v^2},N\right\rangle=0,\quad \left\langle\frac{\partial^2 x}{\partial u\partial v},N\right\rangle=-\beta\cos\varphi(u).$$

Using the fact that $\langle \mathrm{S}X,Y\rangle=\langle \mathrm{d}Y(X),N\rangle$, we compute that the matrix of S in the frame $(\frac{\partial}{\partial u},\frac{\partial}{\partial v})$ is the following:

$$-\frac{\beta\cos\varphi(u)}{\varphi'(u)^2}\begin{pmatrix}0&1\\1&0\end{pmatrix}.$$

In particular the points where $\cos\varphi(u)=0$ are umbilic points. We also have

$$T=\frac{1}{\varphi'(u)^2}\frac{\partial}{\partial v},\quad \nu=\frac{\beta\sin\varphi(u)}{\varphi'(u)}.$$

Remark 4.11. When $\beta=0$, the formula defines a vertical cylinder $\mathbb{S}^1\times\mathbb{R}$. When $\beta\to\infty$, the surface converges to the foliation by horizontal spheres $\mathbb{S}^2\times\{t\}$.

Unduloids. For $\alpha>1$ or $\alpha<-1$, the unduloid $\mathcal{U}_\alpha$ is given by the following conformal immersion:

$$x(u,v)=\begin{pmatrix}\sin\psi(u)\cos\alpha v\\ \sin\psi(u)\sin\alpha v\\ \cos\psi(u)\\ u\end{pmatrix},$$

where the function ψ satisfies

$$1+\psi'(u)^2=\alpha^2\sin^2\psi(u),\quad \psi''(u)=\alpha^2\sin\psi(u)\cos\psi(u). \tag{12}$$

We can assume that $\psi'(0)=0$, $\psi(u)\in(0,\pi)$ and $\cos\psi(0)>0$.

The normal to $\mathcal{U}_\alpha$ in $\mathbb{S}^2\times\mathbb{R}$ is

$$N(u,v)=\frac{1}{\alpha\sin\psi(u)}\begin{pmatrix}-\cos\psi(u)\cos\alpha v\\ -\cos\psi(u)\sin\alpha v\\ \sin\psi(u)\\ \psi'(u)\end{pmatrix}.$$

We compute that the matrix of S in the frame $(\frac{\partial}{\partial u},\frac{\partial}{\partial v})$ is the following:

$$-\frac{\alpha\cos\psi(u)}{1+\psi'(u)^2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

In particular the points where $\cos\psi(u)=0$ are umbilic points. We also have

$$T=\frac{1}{1+\psi'(u)^2}\frac{\partial}{\partial u},\quad \nu=\frac{\psi'(u)}{\alpha\sin\psi(u)}.$$

Remark 4.12. When $\alpha=\pm1$, the formula defines a vertical cylinder $\mathbb{S}^1\times\mathbb{R}$. When $\alpha\to\infty$, the surface converges to the foliation by horizontal spheres $\mathbb{S}^2\times\{t\}$.

Proposition 4.13. *The conjugate surface of the unduloid $\mathcal{U}_\alpha$ is the helicoid $\mathcal{H}_\beta$ with $\alpha^2=1+\beta^2$ and α, β having the same sign.*

Proof. We set $y_1(u) = \alpha \cos\psi(u)$ and $y_2(u) = \beta \cos\varphi(u)$. A computation shows that both y_1 and y_2 are solutions of the following equation:

$$(y')^2 = (y^2 - \alpha^2)(y^2 - \beta^2),$$

and hence of the following equation:

$$y'' = y(2y^2 - \alpha^2 - \beta^2).$$

We have $\psi'(0) = 0$ and so by (12) we have $y_1(0)^2 = \beta^2$ and thus $y_1'(0) = 0$, and $\varphi(0) = 0$ so $y_2(0) = \beta$ and thus $y_2'(0) = 0$. Moreover, $\cos\psi(0) > 0$, so $y_1(0)$ has the sign of α; since α and β have the same sign, we have $y_1(0) = \beta$. By the Cauchy-Lipschitz theorem we conclude that $y_1 = y_2$. From this we deduce using (12) and (11) that $\varphi'(u)^2 = 1 + \psi'(u)^2$, and thus $\mathcal{U}_\alpha$ and $\mathcal{H}_\beta$ are locally isometric, and $\mathrm{S}_{\mathcal{H}_\beta} = \mathrm{J}\mathrm{S}_{\mathcal{U}_\alpha}$ and $T_{\mathcal{H}_\beta} = \mathrm{J}T_{\mathcal{U}_\alpha}$. Finally we have $\nu_{\mathcal{U}_\alpha} = -\frac{y_1'}{\alpha^2 - y_1^2}$ and $\nu_{\mathcal{H}_\beta} = -\frac{y_2'}{\alpha^2 - y_2^2}$, so we get $\nu_{\mathcal{H}_\beta} = \nu_{\mathcal{U}_\alpha}$. □

Remark 4.14. The vertical cylinder $\mathbb{S}^1 \times \mathbb{R}$ is globally invariant by conjugation, but the vertical lines and the horizontal circles are exchanged. For example, a rectangle of height t and whose basis is an arc of angle θ becomes a rectangle of height θ and whose basis is an arc of angle t.

The horizontal sphere $\mathbb{S}^2 \times \{0\}$ is pointwise invariant by conjugation (since it satisfies $\mathrm{S} = 0$ and $T = 0$).

Remark 4.15. The horizontal projections of helicoids and unduloids are the Gauss maps of constant mean curvature Delaunay surfaces in $\mathbb{R}^3$: helicoids in $\mathbb{S}^2 \times \mathbb{R}$ come from nodoids in $\mathbb{R}^3$ and unduloids in $\mathbb{S}^2 \times \mathbb{R}$ come from unduloids in $\mathbb{R}^3$. This correspondance is described in [Ros03].

4.3. **Helicoids and generalized catenoids in $\mathbb{H}^2 \times \mathbb{R}$.** Apart from the horizontal planes $\mathbb{H}^2 \times \{t\}$ and the vertical planes $\mathbb{H}^1 \times \mathbb{R}$ ($\mathbb{H}^1$ being a geodesic of $\mathbb{H}^2$), the most simple examples of minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ are helicoids and catenoids. These surfaces are described in [PR99] and [NR02]. They are properly embedded. Catenoids are rotational; helicoids are invariant by a screw motion and foliated by geodesics of $\mathbb{H}^2$.

More generally, Hauswirth classified minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ foliated by horizontal curves of constant curvature in $\mathbb{H}^2$ ([Hau03]). These surfaces form a two-parameter family. This family includes, among others, catenoids, helicoids and Riemann-type examples. All the surfaces described in this section belong to the Hauswirth family.

Helicoids. For $\beta \neq 0$, the helicoid $\mathcal{H}_\beta$ is given by the following conformal immersion:

$$x(u,v) = \begin{pmatrix} \cosh\varphi(u) \\ \sinh\varphi(u)\cos\beta v \\ \sinh\varphi(u)\sin\beta v \\ v \end{pmatrix},$$

where the function φ satisfies

$$\varphi'(u)^2 = 1 + \beta^2 \sinh^2\varphi(u), \quad \varphi''(u) = \beta^2 \sinh\varphi(u)\cosh\varphi(u). \tag{13}$$

We can assume that $\varphi(0) = 0$ and $\varphi'(u) > 0$. The function φ is defined on a bounded interval. When $\beta > 0$ we say that $\mathcal{H}_\beta$ is a right helicoid; when $\beta < 0$ we say that $\mathcal{H}_\beta$ is a left helicoid.

The normal to $\mathcal{H}_\beta$ in $\mathbb{H}^2 \times \mathbb{R}$ is

$$N(u,v) = \frac{1}{\varphi'(u)} \begin{pmatrix} 0 \\ \sin \beta v \\ -\cos \beta v \\ \beta \sinh \varphi(u) \end{pmatrix}.$$

now $\beta > 0$. we compute that the matrix of S in the frame $(\frac{\partial}{\partial u}, \frac{\partial}{\partial v})$ is the following:

$$-\frac{\beta \cosh \varphi(u)}{\varphi'(u)^2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We also have

$$T = \frac{1}{\varphi'(u)^2} \frac{\partial}{\partial v}, \quad \nu = \frac{\beta \sinh \varphi(u)}{\varphi'(u)}.$$

Remark 4.16. When $\beta = 0$, the fomula defines a vertical plane $\mathbb{H}^1 \times \mathbb{R}$. When $\beta \to \infty$, the surface converges to the foliation by horizontal planes $\mathbb{H}^2 \times \{t\}$.

Catenoids. For $\alpha \neq 0$, the catenoid $\mathcal{C}_\alpha$ is given by the following conformal immersion:

$$x(u,v) = \begin{pmatrix} \cosh \psi(u) \\ \sinh \psi(u) \cos \alpha v \\ \sinh \psi(u) \sin \alpha v \\ u \end{pmatrix},$$

where the function ψ satisfies

$$1 + \psi'(u)^2 = \alpha^2 \sinh^2 \psi(u), \quad \psi''(u) = \alpha^2 \sinh \psi(u) \cosh \psi(u). \tag{14}$$

We can assume that $\psi'(0) = 0$ and $\psi(u) > 0$. The function ψ is defined on the interval $(-u_0, u_0)$ with

$$u_0 = \int_{\psi(0)}^{\infty} \frac{\mathrm{d}\psi}{\sqrt{\alpha^2 \sinh^2 \psi - 1}} = \int_1^\infty \frac{\mathrm{d}x}{\sqrt{(x^2 + \alpha^2)(x^2 - 1)}}.$$

Thus we have

$$u_0 < \int_1^\infty \frac{\mathrm{d}x}{x\sqrt{x^2 - 1}} = \frac{\pi}{2}.$$

This proves that the height of the catenoid $\mathcal{C}_\alpha$ is smaller than π; moreover the height tends to 0 when $\alpha \to \infty$ and to π when $\alpha \to 0$ (theorem 1 in [NR02] holds for $t \in (0, \frac{\pi}{2})$). The function ψ is decreasing on $(-u_0, 0)$ and increasing on $(0, u_0)$. The waist circle is given by $u = 0$.

The normal to $\mathcal{C}_\alpha$ in $\mathbb{H}^2 \times \mathbb{R}$ is

$$N(u,v) = \frac{1}{\alpha \sinh \psi(u)} \begin{pmatrix} -\sinh \psi(u) \\ -\cosh \psi(u) \cos \alpha v \\ -\cosh \psi(u) \sin \alpha v \\ \psi'(u) \end{pmatrix}.$$

We compute that the matrix of S in the frame $(\frac{\partial}{\partial u}, \frac{\partial}{\partial v})$ is the following:

$$-\frac{\alpha \cosh \psi(u)}{1 + \psi'(u)^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We also have

$$T = \frac{1}{1 + \psi'(u)^2} \frac{\partial}{\partial u}, \quad \nu = \frac{\psi'(u)}{\alpha \sinh \psi(u)}.$$

A minimal surface foliated by horocycles. We search a minimal surface such that each horizontal curve is a horocycle in $\mathbb{H}^2$ and such that all the horocycles have the same asymptotic point. Such a surface can be parametrized in the following way:

$$x(u,v) = \begin{pmatrix} \frac{\lambda(u)}{2} + \frac{1+f(u,v)^2}{2\lambda(u)} \\ f(u,v) \\ -\frac{\lambda(u)}{2} + \frac{1+f(u,v)^2}{2\lambda(u)} \\ u \end{pmatrix}$$

with $\lambda > 0$ and $\frac{\partial f}{\partial v} > 0$. This immersion is conformal if and only if

$$\frac{\partial f}{\partial u} = \frac{f\lambda'}{\lambda}, \quad \left(\frac{\partial f}{\partial v}\right)^2 = 1 + \left(\frac{\lambda'}{\lambda}\right)^2.$$

We deduce from the second relation that $\frac{\partial^2 f}{\partial v^2} = 0$, and so

$$f(u,v) = \alpha(u)v + \beta(u).$$

Reporting in the first relation we get

$$\frac{\alpha'}{\alpha} = \frac{\beta'}{\beta} = \frac{\lambda'}{\lambda}.$$

The immersion is minimal if and only if Δx is proportional to the normal $\bar{N}$ to $\mathbb{H}^2 \times \mathbb{R}$; a computation shows that this happens if and only if $(\lambda')^2 + \alpha^2\lambda^2 = \lambda\lambda''$, i.e., if and only if $2(\lambda')^2 + \lambda^2 = \lambda\lambda''$, or, equivalently,

$$\left(\frac{1}{\lambda}\right)'' = -\frac{1}{\lambda}.$$

Up to a reparametrization and an isometry of $\mathbb{H}^2$ we can choose $\lambda(u) = \alpha(u) = \frac{1}{\cos u}$ for $u \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\beta(u) = 0$. Thus we get the following proposition.

Proposition 4.17. *The map*

$$x(u,v) = \begin{pmatrix} \frac{v^2+1}{2\cos u} + \frac{\cos u}{2} \\ \frac{v}{\cos u} \\ \frac{v^2-1}{2\cos u} + \frac{\cos u}{2} \\ u \end{pmatrix}$$

defined for $(u,v) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}$ is a conformal minimal embedding such that the curves $u = u_0$ are horocycles in $\mathbb{H}^2$ having the same asymptotic point. We will denote this surface by $\mathcal{C}_0$.

Morover, the surface $\mathcal{C}_0$ is the unique one (up to isometries of $\mathbb{H}^2 \times \mathbb{R}$) having this property.

In the upper half-plane model for $\mathbb{H}^2$, the curve at height u of $\mathcal{C}_0$ is the horizontal Euclidean line $x_2 = \cos u$. Figure 1 is a picture of $\mathcal{C}_0$ (in this picture the model for $\mathbb{H}^2$ is the Poincaré unit disk model). The surface $\mathcal{C}_0$ has height π. It is symmetric with respect to the horizontal plane $\mathbb{H}^2 \times \{0\}$ and it is invariant by a one-parameter family of horizontal parabolic isometries.

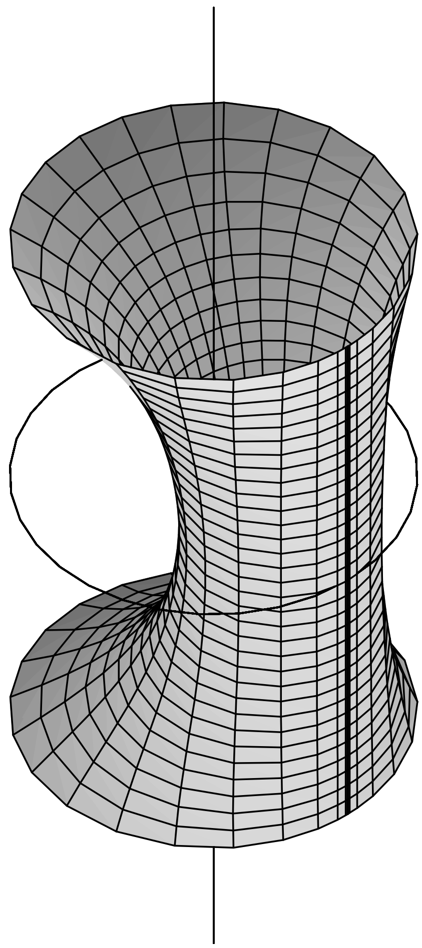

FIGURE 1. A minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ foliated by horocycles.

The normal to $\mathcal{C}_0$ in $\mathbb{H}^2 \times \mathbb{R}$ is

$$N(u,v) = \begin{pmatrix} -\frac{v^2+1}{2} + \frac{\cos^2 u}{2} \\ -v \\ \frac{1-v^2}{2} + \frac{\cos^2 u}{2} \\ \sin u \end{pmatrix}.$$

We compute that the matrix of S in the frame $(\frac{\partial}{\partial u}, \frac{\partial}{\partial v})$ is the following:

$$-\cos u \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We also have

$$T = \cos^2 u \frac{\partial}{\partial u}, \quad \nu = \sin u.$$

Minimal surfaces foliated by equidistants. For $\gamma \in (0,1)$ or $\gamma \in (-1,0)$, we consider the following immersion:

$$x(u,v) = \begin{pmatrix} \cosh\chi(u)\cosh\gamma v \\ \sinh\chi(u) \\ \cosh\chi(u)\sinh\gamma v \\ u \end{pmatrix}.$$

with

$$1+\chi'(u)^2 = \gamma^2\cosh^2\chi(u), \quad \chi''(u) = \gamma^2\cosh\chi(u)\sinh\chi(u). \tag{15}$$

It is a conformal minimal immersion.

We choose χ such that $\chi'(0)=0$ and $\chi(u)>0$. The function χ is defined on the interval $(-u_0,u_0)$ with

$$u_0 = \int_{\chi(0)}^{\infty} \frac{\mathrm{d}\chi}{\sqrt{\gamma^2\cosh^2\chi - 1}} = \int_1^{\infty} \frac{\mathrm{d}x}{\sqrt{(x^2-\gamma^2)(x^2-1)}}.$$

Thus we have

$$u_0 > \int_1^{\infty} \frac{\mathrm{d}x}{x\sqrt{x^2-1}} = \frac{\pi}{2}.$$

We have defined a minimal surface $\mathcal{G}_\gamma$, which we call a generalized catenoid. Its height is greater than π, tends to π when $\gamma \to 0$ and to $+\infty$ when $\gamma \to 1$. The function χ is decreasing on $(-u_0,0)$ and increasing on $(0,u_0)$. The surface is symmetric with respect to the horizontal plane $\mathbb{H}^2 \times \{0\}$ and it is invariant by a one-parameter family of horizontal hyperbolic isometries. The horizontal curves are equidistants to a geodesic in $\mathbb{H}^2$.

The normal to $\mathcal{G}_\gamma$ in $\mathbb{H}^2 \times \mathbb{R}$ is

$$N(u,v) = -\frac{1}{\gamma\cosh\chi(u)} \begin{pmatrix} \sinh\chi(u)\cosh\gamma v \\ \cosh\chi(u) \\ \sinh\chi(u)\sinh\gamma v \\ -\chi'(u) \end{pmatrix}.$$

We compute that the matrix of S in the frame $(\frac{\partial}{\partial u}, \frac{\partial}{\partial v})$ is the following:

$$-\frac{\gamma\sinh\chi(u)}{1+\chi'(u)^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We also have

$$T = \frac{1}{1+\chi'(u)^2}\frac{\partial}{\partial u}, \quad \nu = \frac{\chi'(u)}{\gamma\cosh\chi(u)}.$$

Remark 4.18. When $\gamma = \pm 1$, the formula defines a vertical plane $\mathbb{H}^1 \times \mathbb{R}$.

Proposition 4.19. *The conjugate surface of the catenoid $\mathcal{C}_\alpha$ is the helicoid $\mathcal{H}_\beta$ with $\beta^2 = 1+\alpha^2$ and α, β having the same sign.*

Proof. We set $y_1(u) = \alpha\cosh\psi(u)$ and $y_2(u) = \beta\cosh\varphi(u)$. A computation shows that both y_1 and y_2 are solutions of the following equation:

$$(y')^2 = (y^2-\alpha^2)(y^2-\beta^2),$$

and hence of the following equation:

$$y'' = y(2y^2-\alpha^2-\beta^2).$$

We have $\psi'(0) = 0$ and so by (14) we have $y_1(0)^2 = \beta^2$ and thus $y_1'(0) = 0$, and $\varphi(0) = 0$ so $y_2(0) = \beta$ and thus $y_2'(0) = 0$. Moreover, $y_1(0)$ has the sign of α, i.e., the sign of β, so we get $y_1(0) = \beta$. By the Cauchy-Lipschitz theorem we conclude that $y_1 = y_2$ (and in particular they have the same domain of definition). From this we deduce using (14) and (13) that $\varphi'(u)^2 = 1 + \psi'(u)^2$, and thus $\mathcal{C}_\alpha$ and $\mathcal{H}_\beta$ are locally isometric, $\mathrm{S}_{\mathcal{H}_\beta} = \mathrm{JS}_{\mathcal{C}_\alpha}$ and $T_{\mathcal{H}_\beta} = \mathrm{J}T_{\mathcal{C}_\alpha}$. Finally we have $\nu_{\mathcal{C}_\alpha} = \frac{y_1'}{y_1^2-\alpha^2}$ and $\nu_{\mathcal{H}_\beta} = \frac{y_2'}{y_2^2-\alpha^2}$, so we get $\nu_{\mathcal{H}_\beta} = \nu_{\mathcal{C}_\alpha}$. □

Proposition 4.20. *The conjugate surface of the surface $\mathcal{C}_0$ is the helicoid $\mathcal{H}_1$.*

Proof. In the case where $\beta = 1$, the function φ satisfies $\varphi' = \cosh\varphi$, and thus we have $\varphi(u) = \ln(\tan(\frac{u}{2} + \frac{\pi}{4}))$, $\varphi'(u) = \frac{1}{\cos u}$ and $\sinh\varphi(u) = \tan u$. Then, using the above calculations, we easily check that $\mathcal{C}_0$ and $\mathcal{H}_1$ are locally isometric, and that $\mathrm{S}_{\mathcal{H}_1} = \mathrm{JS}_{\mathcal{C}_0}$, $T_{\mathcal{H}_1} = \mathrm{J}T_{\mathcal{C}_0}$, $\nu_{\mathcal{H}_1} = \nu_{\mathcal{C}_0}$. □

Remark 4.21. The conjugate surface of the surface $\mathcal{C}_0$ with the opposite orientation is the helicoid $\mathcal{H}_{-1}$.

Proposition 4.22. *The conjugate surface of the generalized catenoid $\mathcal{G}_\gamma$ is the helicoid $\mathcal{H}_\beta$ with $\beta^2 + \gamma^2 = 1$ and β, γ having the same sign.*

Proof. We set $y_1(u) = \gamma\sinh\chi(u)$ and $y_2(u) = \beta\cosh\varphi(u)$. A computation shows that both y_1 and y_2 are solutions of the following equation:

$$(y')^2 = (y^2 + \gamma^2)(y^2 - \beta^2),$$

and hence of the following equation:

$$y'' = y(2y^2 + \gamma^2 - \beta^2).$$

We have $\chi'(0) = 0$ and so by (15) we have $y_1(0)^2 = \beta^2$ and thus $y_1'(0) = 0$, and $\varphi(0) = 0$ so $y_2(0) = \beta$ and thus $y_2'(0) = 0$. Moreover, $y_1(0)$ has the sign of γ, i.e., the sign of β, so we get $y_1(0) = \beta$. By the Cauchy-Lipschitz theorem we conclude that $y_1 = y_2$ (and in particular they have the same domain of definition). From this we deduce using (15) and (13) that $\varphi'(u)^2 = 1 + \chi'(u)^2$, and thus $\mathcal{G}_\gamma$ and $\mathcal{H}_\beta$ are locally isometric, $\mathrm{S}_{\mathcal{H}_\beta} = \mathrm{JS}_{\mathcal{G}_\gamma}$ and $T_{\mathcal{H}_\beta} = \mathrm{J}T_{\mathcal{G}_\gamma}$. Finally we have $\nu_{\mathcal{G}_\gamma} = \frac{y_1'}{y_1^2+\gamma^2}$ and $\nu_{\mathcal{H}_\beta} = \frac{y_2'}{y_2^2+\gamma^2}$, so we get $\nu_{\mathcal{H}_\beta} = \nu_{\mathcal{G}_\gamma}$. □

Remark 4.23. This study shows that there are three types of helicoid conjugates according to the parameter of the screw-motion associated to the helicoid: the first type ones are the catenoids, which are rotational surfaces, the second type one is $\mathcal{C}_0$, which is invariant by a one-parameter family of horizontal parabolic isometries and which corresponds to a critical value of the parameter, the third type ones are the generalized catenoids, which are invariant a one-parameter family of horizontal hyperbolic isometries.

This phenomenon is very similar to what happens for the conjugate cousins in $\mathbb{H}^3$ of the helicoids in $\mathbb{R}^3$. There exists an isometric correspondance between minimal surfaces in $\mathbb{R}^3$ and constant mean curvature one surfaces in $\mathbb{H}^3$ called the cousin relation (see [Bry87] and [UY93]). Starting from a helicoid in $\mathbb{R}^3$, we consider its conjugate surface, which is a catenoid in $\mathbb{R}^3$, and then the cousin surface in $\mathbb{H}^3$, which is a catenoid cousin. Catenoid cousins are of three types according to the parameter of the minimal helicoid: some are rotational surfaces, one is invariant by

a one-parameter family of parabolic isometries (and corresponds to a critical value of the parameter), some are invariant by a one-parameter family of hyperbolic isometries. These surfaces are described in details in [SET01] and [Ros02a].

Remark 4.24. All the above surfaces belong to the Hauswirth family: with the notations of [Hau03], helicoids correspond to $d = 0$, $c > 0$, $c \neq 1$; catenoids correspond to $c = 0$, $d > 1$; the surface $\mathcal{C}_0$ corresponds to $c = 0$, $d = 1$; the surfaces $\mathcal{G}_\gamma$ correspond to $c = 0$, $d \in (0, 1)$.

Remark 4.25. The vertical plane $\mathbb{H}^1 \times \mathbb{R}$ is globally invariant by conjugation, but the vertical lines and the horizontal geodesics of $\mathbb{H}^2$ are exchanged. The horizontal plane $\mathbb{H}^2 \times \{0\}$ is pointwise invariant by conjugation (since it satisfies $\mathrm{S} = 0$ and $T = 0$). This is similar to what happens in $\mathbb{S}^2 \times \mathbb{R}$.

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